Introduction

This document is a work-in-progress solution manual for Tom Apostol’s *Introduction to Analytic Number Theory*. The solutions were worked out primarily for my learning of the subject, as Cornell University currently does not offer an analytic number theory course at either the undergraduate or graduate level. However, this document is public and available for use by anyone. If you are a student using this document for a course, I recommend that you first try work out the problems by yourself or in a group. My math documents are stored on a math blog at www.epicmath.org.

3 Averages of Arithmetical Functions

3.1. Use Euler’s summation formula to deduce the following for $x \geq 2$.

\[
\sum_{n \leq x} \frac{\log n}{n} = \frac{1}{2} \log^2 x + A + O\left(\frac{\log x}{x}\right), \text{ where } A \text{ is a constant.}
\]

*Proof.* Note that

\[
\frac{d}{dt} \frac{\log t}{t} = \frac{1 - \log t}{t^2}.
\]
By Euler’s summation formula we have

\[
\sum_{n \leq x} \frac{\log n}{n} = \int_1^x \frac{\log t}{t} dt + \int_1^x (t - [t]) \frac{1 - \log t}{t^2} dt + \frac{\log x}{x} ([x] - x)
\]

\[
= \frac{1}{2} \log^2 x - \int_1^\infty (t - [t]) \frac{1 - \log t}{t^2} dt + \int_x^\infty (t - [t]) \frac{1 - \log t}{t^2} dt
\]

\[+ O\left(\frac{\log x}{x}\right).\]

Note that the remainder results from the fact that \([x] - x\) is bounded by \(-1\) and 0. Now

\[
\int_1^\infty (t - [t]) \frac{1 - \log t}{t^2} dt = \int_1^\infty (t - [t]) \frac{1}{t^2} dt - \int_1^\infty (t - [t]) \frac{\log t}{t^2} dt,
\]

which are bounded by

\[
\int_1^\infty (t - [t]) \frac{1}{t^2} dt \leq \int_1^\infty \frac{1}{t^2} dt = 1
\]

and

\[
\int_x^\infty (t - [t]) \frac{\log t}{t^2} dt \leq \int_x^\infty \frac{\log t}{t^2} dt = \log x \frac{1}{x}
\]

so that the term converges and is equal to a constant, call it \(A\). Similarly for the \(\int_x^\infty (t - [t]) \frac{1 - \log t}{t^2} dt\) term, we have

\[
\int_x^\infty (t - [t]) \frac{1}{t^2} dt \leq \int_x^\infty \frac{1}{t^2} dt = \frac{1}{x}
\]

and

\[
\int_1^\infty (t - [t]) \frac{\log t}{t^2} dt \leq \int_1^\infty \frac{\log t}{t^2} dt = \frac{\log x + 1}{x}
\]

Hence \(\int_x^\infty (t - [t]) \frac{1 - \log t}{t^2} dt = O\left(\frac{\log x}{x}\right)\) and since \(O\left(\frac{\log x}{x}\right) + O\left(\frac{\log x}{x}\right) = O\left(\frac{\log x}{x}\right)\), we have

\[
\sum_{n \leq x} \frac{\log n}{n} = \frac{1}{2} \log^2 x + A + O\left(\frac{\log x}{x}\right).
\]

\[\text{(b)}\ \sum_{2 \leq n \leq x} \frac{1}{n \log n} = \log(\log x) + B + O\left(\frac{1}{x \log x}\right), \text{ where } B \text{ is a constant.}
\]

\[\text{Proof.} \text{ The derivative is}
\]

\[
\frac{d}{dt} \frac{1}{t \log t} = -\frac{1 + \log t}{t^2 \log^2 t}.
\]
By Euler’s summation formula we have

\[
\sum_{2 \leq n \leq x} \frac{1}{n \log n} = \int_2^x \frac{dt}{t \log t} - \int_2^x (t - [t]) \frac{1 + \log t}{t^2 \log^2 t} dt + \frac{1}{x \log x} ([x] - x)
\]

\[
= \log(\log x) - \log(\log 2) + \int_2^\infty (t - [t]) \frac{1 + \log t}{t^2 \log^2 t} dt
\]

\[
- \int_x^\infty (t - [t]) \frac{1 + \log t}{t^2 \log^2 t} dt + O \left( \frac{1}{x \log x} \right).
\]

Now the first integral term is bounded and hence a constant as

\[
\int_2^\infty (t - [t]) \frac{1 + \log t}{t^2 \log^2 t} dt \leq \int_2^\infty \frac{1 + \log t}{t^2 \log^2 t} dt = \frac{1}{2 \log 2},
\]

and the second integral term follows

\[
\int_x^\infty (t - [t]) \frac{1 + \log t}{t^2 \log^2 t} dt \leq \int_x^\infty \frac{1 + \log t}{t^2 \log^2 t} dt = \frac{1}{x \log x} = O \left( \frac{1}{x \log x} \right).
\]

Hence the sum is equal to

\[
\sum_{2 \leq n \leq x} \frac{1}{n \log n} = \log(\log x) + B + O \left( \frac{1}{x \log x} \right).
\]

3.2. If \( x \geq 2 \) prove that

\[
\sum_{n \leq x} \frac{\sigma(n)}{n} = \frac{1}{2} \log^2 x + 2C \log x + O(1), \text{ where } C \text{ is Euler’s constant.}
\]
Proof. Let \( n = qd \). Then using Theorem 3.2(a) and Exercise 3.1(a), we have

\[
\sum_{n \leq x} \frac{d(n)}{n} = \sum_{n \leq x} \sum_{d \mid n} \frac{1}{n} = \sum_{q, d \leq x} \frac{1}{qd} = \sum_{d \leq x} \frac{1}{d} \sum_{q \leq \frac{x}{d}} \frac{1}{q} = \sum_{d \leq x} \frac{1}{d} \left\{ \log x - \log d + C + O \left( \frac{1}{x} \right) \right\} = \left\{ \log x + C + O \left( \frac{1}{x} \right) \right\} \left\{ \log x + C + O \left( \frac{1}{x} \right) \right\} - \sum_{d \leq x} \frac{\log d}{d}
\]

\[
= \log^2 x + 2C \log x + C^2 + 2O \left( \frac{1}{x} \right) (\log x + C) - \frac{1}{2} \log^2 x - A + O \left( \frac{\log x}{x} \right)
\]

\[
= \frac{1}{2} \log^2 x + 2C \log x + O(1).
\]

3.3. If \( x \geq 2 \) and \( \alpha < 0, \alpha \neq 1 \), prove that

\[
\sum_{n \leq x} \frac{d(n)}{n^\alpha} = \frac{x^{1-\alpha} \log x}{1-\alpha} + \zeta(\alpha)^2 + O(x^{1-\alpha}).
\]
Proof. Let $n = qd$. Using Theorem 3.2(b), we have

$$\sum_{n \leq x} \frac{d(n)}{n^{\alpha}} = \sum_{n \leq x} \sum_{d|n} \frac{1}{n^{\alpha}}$$

$$= \sum_{q,d \leq x} \sum_{d|n} \frac{1}{(qd)^{\alpha}}$$

$$= \sum_{d \leq x} \frac{1}{d^{\alpha}} \sum_{q \leq \frac{x}{d}} \frac{1}{q^{\alpha}}$$

$$= \sum_{d \leq x} \frac{1}{d^{\alpha}} \left\{ \frac{x^{1-\alpha}}{1-\alpha} \cdot \frac{1}{d^{x-\alpha}} + \zeta(\alpha) + O(x^{-\alpha}) \right\}$$

$$= \frac{x^{1-\alpha}}{1-\alpha} \left\{ \log x + C + O \left( \frac{1}{x} \right) \right\}$$

$$+ \left\{ \frac{x^{1-\alpha}}{1-\alpha} + \zeta(\alpha) + O(x^{-\alpha}) \right\} \left\{ \zeta(\alpha) + O(x^{-\alpha}) \right\}$$

$$= \frac{x^{1-\alpha}}{1-\alpha} \log x + C \cdot \frac{x^{1-\alpha}}{1-\alpha} + O \left( \frac{x^{-\alpha}}{1-\alpha} \right)$$

$$+ \frac{x^{1-\alpha}}{1-\alpha} \zeta(\alpha) + \zeta(\alpha)^2 + 2\zeta(\alpha)O(x^{-\alpha}) + O(x^{-2\alpha})$$

$$= \frac{x^{1-\alpha}}{1-\alpha} \log x + \zeta(\alpha)^2 + O \left( \frac{x^{1-\alpha}}{1-\alpha} \right)$$

3.4. If $x \geq 2$ prove that:

(a) $\sum_{n \leq x} \mu(n) \left[ \frac{x}{n} \right]^2 = \frac{x^2}{\zeta(2)} + O(x \log x)$.

Proof. Let $g = 2N - u$, so that $g(n) = 2n - 1$ and $G(x) = \sum_{n \leq x} g(n) = \lfloor x \rfloor^2$. By Theorems 3.10 and 3.7, we have

$$\sum_{n \leq x} \mu(n) \left[ \frac{x}{n} \right]^2 = \sum_{n \leq x} (\mu * (2N - u))(n)$$

$$= \sum_{n \leq x} (2\varphi(n) - I(n))$$

$$= \frac{6}{\pi^2} x^2 + O(x \log x) - 1$$

$$= \frac{1}{\zeta(2)} x^2 + O(x \log x).$$

(b) $\sum_{n \leq x} \frac{\mu(n) \lfloor x \rfloor}{n} = \frac{x}{\zeta(2)} + O(\log x).$
Proof. Note that
\[
\left\lfloor \frac{x}{n} \right\rfloor \cdot \frac{1}{x} = \left( \frac{x}{n} + O(1) \right) \cdot \frac{1}{x} = \frac{1}{n} + O \left( \frac{1}{x} \right).
\]
Thus by part (a) and Theorem 3.12, we have
\[
\sum_{n \leq x} \frac{1}{n} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor = \sum_{n \leq x} \left\{ \frac{x}{n} \cdot \frac{1}{x} + O \left( \frac{1}{x} \right) \right\} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor
= \frac{1}{x} \sum_{n \leq x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor^2 + O \left( \frac{1}{x} \right) \sum_{n \leq x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor
= \frac{1}{x} \left( \frac{x^2}{\zeta(2)} + O(x \log x) \right) + O \left( \frac{1}{x} \right) \cdot 1
= \frac{x}{\zeta(2)} + O(\log x).
\]

3.5. If \( x \geq 1 \) prove that:

(a) \( \sum_{n \leq x} \varphi(n) = \frac{1}{2} \sum_{n \leq x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor^2 + \frac{1}{2} \)

Proof. Recall from the proof of Exercise 3.4(a) that
\[
\sum_{n \leq x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor^2 = \sum_{n \leq x} (2\varphi(n) - I(n)).
\]
Since \( \sum_{n \leq x} I(n) = 1 \), rearranging gives
\[
\sum_{n \leq x} \varphi(n) = \frac{1}{2} \sum_{n \leq x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor^2 + \frac{1}{2}.
\]

(b) \( \sum_{n \leq x} \frac{\varphi(n)}{n} = \sum_{n \leq x} \frac{\mu(n)}{n} \left\lfloor \frac{x}{n} \right\rfloor \)

Proof. Let \( g(n) = 1 \), then \( G(x) = \sum_{n \leq x} g(n) = \lfloor x \rfloor \). By Theorem 3.10, we have
\[
\sum_{n \leq x} \frac{\mu(n)}{n} \left\lfloor \frac{x}{n} \right\rfloor = \sum_{n \leq x} ((N^{-1} \mu) * u)(n)
= \sum_{n \leq x} \sum_{d \mid n} \frac{\mu(d)}{d}
= \sum_{n \leq x} \frac{1}{n} \sum_{d \mid n} \mu(d) \frac{n}{d}
= \sum_{n \leq x} \frac{1}{n} \varphi(n).
\]
3.6. If \( x \geq 2 \) prove that

\[
\sum_{n \leq x} \frac{\varphi(n)}{n^2} = \frac{1}{\zeta(2)} \log x + \frac{C}{\zeta(2)} - A + O\left(\frac{\log x}{x}\right),
\]

where \( C \) is Euler’s constant and

\[
A = \sum_{1}^{\infty} \frac{\mu(n) \log n}{n^2}.
\]

Proof. Note that \( N^{-2} \) is completely multiplicative, so

\[
n^{-2} \varphi(n) = (N^{-2} \cdot (\mu \ast N))(n) = ((N^{-2} \cdot \mu) \ast (N^{-1}))(n).
\]

Then by Theorem 3.10 and 3.2(a), we have

\[
\sum_{n \leq x} \frac{\varphi(n)}{n^2} = \sum_{n \leq x} ((N^{-2} \cdot \mu) \ast (N^{-1}))(n)
\]

\[
= \sum_{n \leq x} \frac{\mu(n)}{n^2} \left\{ \log \left(\frac{x}{n}\right) + C + O\left(\frac{1}{x}\right) \right\}
\]

\[
= \left\{ \frac{6}{\pi^2} + O\left(\frac{1}{x}\right) \right\} \left\{ \log x + C + O\left(\frac{1}{x}\right) \right\} - \sum_{n \leq x} \frac{\mu(n) \log n}{n^2}
\]

\[
= \frac{1}{\zeta(2)} \log x + \frac{C}{\zeta(2)} + O\left(\frac{\log x}{x}\right) - \sum_{n \leq x} \frac{\mu(n) \log n}{n^2}.
\]

Now

\[
\sum_{n \leq x} \frac{\mu(n) \log n}{n^2} = \sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n^2} - \sum_{x > n} \frac{\mu(n) \log n}{n^2}.
\]

The sum from 1 to infinity is clearly finite, and the other sum satisfies

\[
\left| \sum_{x > n} \frac{\mu(n) \log n}{n^2} \right| \leq \sum_{x > n} \frac{\log n}{n^2} = O\left(\frac{\log x}{x}\right),
\]

hence

\[
\sum_{n \leq x} \frac{\varphi(n)}{n^2} = \frac{1}{\zeta(2)} \log x + \frac{C}{\zeta(2)} - A + O\left(\frac{\log x}{x}\right).
\]
3.7. In a later chapter we will prove that $\sum_{n=1}^{\infty} \mu(n)n^{-\alpha} = 1/\zeta(\alpha)$ if $\alpha > 1$. Assuming this, prove that for $x \geq 2$ and $\alpha > 1$, $\alpha \neq 2$, we have

$$\sum_{n \leq x} \frac{\varphi(n)}{n^\alpha} = \frac{x^{2-\alpha}}{2 - \alpha} \frac{1}{\zeta(2)} + \frac{\zeta(\alpha - 1)}{\zeta(\alpha)} + O(x^{1-\alpha} \log x).$$

**Proof.** Note that $N^{-\alpha}$ is completely multiplicative, so

$$n^{-\alpha} \varphi(n) = (N^{-\alpha} \cdot (\mu * N))(n) = ((N^{-\alpha} \cdot \mu) \ast (N^{1-\alpha}))(n).$$

Also, note that

$$\sum_{n \leq x} \frac{\mu(n)}{n^\alpha} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^\alpha} - \sum_{n>x} \frac{\mu(n)}{n^\alpha},$$

where the first sum is assumed to be $1/\zeta(\alpha)$ and the second sum satisfies

$$\left| \sum_{n>x} \frac{\mu(n)}{n^\alpha} \right| \leq \sum_{n>x} \frac{1}{n^\alpha} = O(x^{1-\alpha}).$$

Then by Theorem 3.10 and 3.2(b), we have

$$\sum_{n \leq x} \frac{\varphi(n)}{n^\alpha} = \sum_{n \leq x} ((N^{-\alpha} \cdot \mu) \ast (N^{1-\alpha}))(n)$$

$$= \sum_{n \leq x} \frac{\mu(n)}{n^\alpha} \left( \frac{\left( \frac{x}{n} \right)^{2-\alpha}}{2 - \alpha} + \frac{\zeta(\alpha - 1)}{\zeta(\alpha)} + O \left( \frac{\left( \frac{x}{n} \right)^{1-\alpha}}{\zeta(\alpha)} \right) \right)$$

$$= \frac{x^{2-\alpha}}{2 - \alpha} \sum_{n \leq x} \frac{\mu(n)}{n^\alpha} + \left( \frac{\zeta(\alpha - 1)}{\zeta(\alpha)} + O(1) \right) + O \left( \frac{x^{1-\alpha}}{\zeta(\alpha)} \right)$$

$$= \frac{x^{2-\alpha}}{2 - \alpha} \frac{1}{\zeta(2)} + \frac{\zeta(\alpha - 1)}{\zeta(\alpha)} + O(x^{1-\alpha} \log x).$$

3.8. If $\alpha \leq 1$ and $x \geq 2$ prove that

$$\sum_{n \leq x} \frac{\varphi(n)}{n^\alpha} = \frac{x^{2-\alpha}}{2 - \alpha} \frac{1}{\zeta(2)} + O(x^{1-\alpha} \log x).$$

**Proof.** The proof is the same as for Exercise 3.7 except that due to the condition on $\alpha$, we use Theorem 3.2(d) instead of Theorem 3.2(b).

3.9. In a later chapter we will prove that the infinite product $\prod_p (1 - p^{-2})$, extended over all primes, converges to the value $1/\zeta(2) = 6/\pi^2$. Assuming this result, prove that
(a) \( \frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} < \frac{n^2 \sigma(n)}{6n} \) if \( n \geq 2 \).

Proof. Let \( n = \prod_i a_i^{p_i} \). Then from the formula on p.39 and using the fact that \( \sigma \) is multiplicative, we have

\[
\frac{\sigma(n)}{n} = \frac{1}{n} \prod_i \frac{p_i^a_i + 1}{p_i - 1}.
\]

Now we also have

\[
\frac{n}{\varphi(n)} = \prod_i \frac{p_i}{p_i - 1}.
\]

Dividing gives

\[
\frac{n/\varphi(n)}{\sigma(n)/n} = \prod_i \frac{p_i}{p_i - 1} \frac{p_i^a_i + 1}{p_i^a_i + 1 - 1} = \prod_i \left( \frac{p_i^a_i + 1}{p_i^a_i + 1 - 1} \right)^{-1} = \prod_i \left( 1 - \frac{1}{p_i^a_i + 1} \right)^{-1}.
\]

The product is clearly minimized when \( n = 1 \), i.e. there are no prime factors. To maximize the product, we need to have as many prime factors as possible and minimize the \( a_i \). Since \( a_i \geq 1 \) by our construction of the prime factorization, we have that the maximum is

\[
\prod_p \left( 1 - \frac{1}{p^2} \right)^{-1} = \zeta(2) = \pi^2/6.
\]

This proves both inequalities.

(b) If \( x \geq 2 \) prove that

\[
\sum_{n \leq x} \frac{n}{\varphi(n)} = O(x).
\]

Proof. From part (a) we know that

\[
\sum_{n \leq x} \frac{n}{\varphi(n)} = O \left( \sum_{n \leq x} \frac{\sigma(n)}{n} \right),
\]

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thus it suffices to show \( \sum_{n \leq x} \frac{\sigma(n)}{n} = O(x) \). In fact, we show the asymptotic behavior \( \sum_{n \leq x} \frac{\sigma(n)}{n} = \zeta(2)x + O(\log x) \). Let \( n = qd \), then

\[
\sum_{n \leq x} \frac{\sigma(n)}{n} = \sum_{d \leq x} \sum_{q \leq x/d} \frac{1}{d} = \sum_{d \leq x} \frac{1}{d} \left\lfloor \frac{x}{d} \right\rfloor \]
\[
= \sum_{d \leq x} \frac{1}{d} \left( \frac{x}{d} + O(1) \right) = x \sum_{d \leq x} \frac{1}{d^2} + \sum_{d \leq x} \frac{1}{d} O(1)
\]
\[
= x \left( -\frac{1}{x} + \zeta(2) + O(x^{-2}) \right) + O(\log x) = \zeta(2)x + O(\log x).
\]

3.10. If \( x \geq 2 \) prove that

\[
\sum_{n \leq x} \frac{1}{\varphi(n)} = O(\log x).
\]

**Proof.** In Exercise 3.9(a) we showed that for \( n \geq 2 \), \( \frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} < \frac{x^2}{6} \frac{\sigma(n)}{n} \). Hence it suffices to show

\[
\sum_{n \leq x} \frac{\sigma(n)}{n^2} = O(\log x).
\]

Let \( n = qd \), then

\[
\sum_{n \leq x} \frac{\sigma(n)}{n^2} = \sum_{d \leq x} \sum_{q \leq x/d} \frac{1}{qd^2} = \sum_{d \leq x} \frac{1}{d^2} \sum_{q \leq x/d} \frac{1}{d} \log \frac{x}{q}
\]
\[
= \sum_{d \leq x} \frac{1}{d^2} \left( \log \frac{x}{d} + C + O\left( \frac{1}{x} \right) \right)
\]
\[
= \left( \log x + C + O\left( \frac{1}{x} \right) \right) \sum_{d \leq x} \frac{1}{d^2} - \sum_{d \leq x} \frac{\log d}{d^2} = \left( \log x + C + O\left( \frac{1}{x} \right) \right) \left( -\frac{1}{x} + \zeta(2) + O(x^{-2}) \right) + O\left( \frac{\log x}{x} \right)
\]
\[
= \zeta(2) \log x + \zeta(2) C + O\left( \frac{\log x}{x} \right).
\]
Thus $\sum_{n \leq x} \frac{\sigma(n)}{n} = O(\log x)$ and we are done.

3.11. Let $\varphi_1(n) = n \sum_{d|n} |\mu(d)|/d$.

(a) Prove that $\varphi_1$ is multiplicative and that $\varphi_1(n) = n \prod_{p|n}(1 + p^{-1})$.

Proof. Let $m$ and $n$ be positive integers such that $(m, n) = 1$. Then

$$\varphi_1(m)\varphi_1(n) = \left( m \sum_{d|m} \frac{|\mu(d)|}{d} \right) \left( n \sum_{d|n} \frac{|\mu(d)|}{d} \right) = mn \sum_{d|mn} \frac{|\mu(d)|}{d} = \varphi_1(mn),$$

hence $\varphi_1$ is multiplicative. Plugging in prime powers shows

$$\sum_{d|p^k} \frac{|\mu(d)|}{d} = 1 + 1/p = \prod_{p|n}(1 + 1/p).$$

(b) Prove that

$$\varphi_1(n) = \sum_{d^2|n} \mu(d)\sigma\left(\frac{n}{d^2}\right)$$

where the sum is over the divisors of $n$ for which $d^2|n$.

Proof. Since $\varphi_1$ is multiplicative it suffices to show this for prime powers $p^k$. The product definition in part (a) yields

$$\varphi_1(p^k) = p^k \prod_{p|n}(1 + p^{-1}) = p^k + p^{k-1}.$$  

The formula in part (b) yields

$$\sum_{d^2|n} \mu(d)\sigma\left(\frac{n}{d^2}\right) = \sigma(p^k) - \sigma(p^{k-2})$$

$$= \frac{(p^{k+1} - 1) - (p^{k-1} - 1)}{p - 1}$$

$$= \frac{p^{k-1}p^2 - 1}{p - 1}$$

$$= \frac{p^{k-1}(p + 1)}{p - 1}$$

$$= p^k + p^{k-1},$$

so that the two definitions are the same.

(c) Prove that

$$\sum_{n \leq x} \varphi_1(n) = \sum_{d \leq \sqrt{x}} \mu(d)S\left(\frac{x}{d^2}\right),$$

where $S(x) = \sum_{k \leq x} \sigma(k)$,
then use Theorem 3.4 to deduce that, for $x \geq 2$, 
\[
\sum_{n \leq x} \varphi_1(n) = \frac{\zeta(2)}{2\zeta(4)} x^2 + O(x \log x).
\]

As in Exercise 3.7, you may assume the result $\sum_{n=1}^{\infty} \mu(n)n^{-\alpha} = 1/\zeta(\alpha)$ for $\alpha > 1$.

Proof. The proof of the first statement follows easily from using Theorem 3.10 on the result in part (b). The second statement can be shown as follows:

\[
\sum_{n \leq x} \varphi_1(n) = \sum_{d \leq \sqrt{x}} \mu(d) S \left( \frac{x}{d^2} \right)
\]
\[
= \sum_{d \leq \sqrt{x}} \mu(d) \left( \frac{1}{2} \zeta(2) \left( \frac{x}{d^2} \right)^2 + O \left( \frac{x}{d^2} \log \frac{x}{d^2} \right) \right)
\]
\[
= \frac{1}{2} \zeta(2) x^2 \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^4} + O \left( x \log x \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^4} - 2x \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^2} \log d \right)
\]
\[
= \frac{\zeta(2)}{2\zeta(4)} x^2 + O(x \log x).
\]

3.12. For real $s > 0$ and integer $k \geq 1$ find an asymptotic formula for the partial sums

\[\sum_{n \leq x} \frac{1}{n^s}\]

with an error term that tends to 0 as $x \to \infty$. Be sure to include the case $s = 1$.

Proof. Note that $(n, k) = 1$ imply that $n$ and $k$ share no common prime factors. Hence, for each prime factor $p$ of $k$, we need to subtract from $\sum_{n \leq x} 1/n^s$ those $\sum_{n \leq x, p|n} 1/n^s$. For example, if $k = 2$ then we have

\[
\sum_{(n, 2) = 1} \frac{1}{n^s} = \sum_{n \leq x} \frac{1}{n^s} - \sum_{n \leq x} \frac{1}{n^s}
\]
\[
= \sum_{n \leq x} \frac{1}{n^s} - \sum_{n \leq x/2} \frac{1}{n^s}
\]
\[
= \sum_{n \leq x} \frac{1}{n^s} - \frac{1}{2} \sum_{n \leq x/2} \frac{1}{n^s}.
\]
Now we need to be careful when there are multiple prime factors. For example, if \( k = 6 \), then when we subtract the multiples of 2 and multiples of 3, we have ended up subtracting the multiples of 6 twice, so we need to add back the multiples of 6. The function that takes on 1 at 1, \(-1\) at primes, 1 at the product of two primes, etc. is the Möbius function, \( \mu \). Hence we may write the sum as

\[
\sum_{n \leq x} \frac{1}{n^s} = \sum_{d | k} \frac{\mu(d)}{d^s} \sum_{n \leq x/d} \frac{1}{n^s}.
\]

Now we need to consider the two cases \( s = 1 \) and \( s \neq 1 \). If \( s = 1 \) then we have

\[
\sum_{d | k} \frac{\mu(d)}{d} \sum_{n \leq x/d} \frac{1}{n} = \sum_{d | k} \frac{\mu(d)}{d} \left( \log x - \log d + C + O \left( \frac{d}{x} \right) \right)
= \log x \sum_{d | k} \frac{\mu(d)}{d} + C \sum_{d | k} \frac{\mu(d)}{d} - \sum_{d | k} \frac{\mu(d) \log d}{d} + O \left( \frac{1}{x} \right),
\]

where \( \sum_{d | k} \mu(d)/d \) and \( \sum_{d | k} (\mu(d) \log d)/d \) are constants based on \( k \). If \( s \neq 1 \),

\[
\sum_{d | k} \frac{\mu(d)}{d^s} \sum_{n \leq x/d} \frac{1}{n^s} = \sum_{d | k} \frac{\mu(d)}{d^s} \left( \frac{x^{1-s}}{1-s} \frac{1}{d^{1-s}} + \zeta(s) + O \left( \frac{x^{s-1}}{d^s} \right) \right)
= \frac{x^{1-s}}{1-s} \sum_{d | k} \frac{\mu(d)}{d} + \zeta(s) \sum_{d | k} \frac{\mu(d)}{d^s} + O(x^{-s}),
\]

where \( \sum_{d | k} \mu(d)/d \) and \( \sum_{d | k} \mu(d)/d^s \) are constants based on \( k \) and \( s \).

3.13–3.26. For each real \( x \) the symbol \([x]\) denotes the greatest integer \( \leq x \). Exercises 13 through 26 describe some properties of the greatest-integer function. In these exercises \( x \) and \( y \) denote real numbers, \( n \) denotes an integer.

3.13. Prove each of the following statements.

(a) If \( x = k + y \) where \( k \) is an integer and \( 0 \leq y < 1 \), then \( k = [x] \).

Proof. \( x - y = k \) is an integer, and since \( 0 \leq y < 1 \), \( k \) is also the greatest integer \( \leq x \).

(b) \([x + n] = [x] + n\).

Proof. Let \( x = k + y \), where \( k \) is an integer and \( 0 \leq y < 1 \). Then by part (a), we have

\([x + n] = [y + k + n] = k + n = [x] + n\).

(c) \([-x] = \begin{cases} [-x] & \text{if } x = [x], \\ -[x] - 1 & \text{if } x \neq [x]. \end{cases}\)
Proof. If \( x = [x] \) the statement is trivial. Else, let \( x = k + y \), where \( k \) is an integer and \( 0 < y < 1 \). Then
\[
[-x] = [-k - y] = [-k - 1 + (1 - y)] = -k - 1 = -[x] - 1.
\]
(d) \([x/n] = [[x]/n]\) if \( n \geq 1 \).

Proof. Let \( x = k + y \), where \( k \) is an integer and \( 0 \leq y < 1 \). Then let \( k/n = m + z \), where \( m \) is an integer and \( 0 \leq z < 1 \). We have
\[
[x/n] = [(k + y)/n] = [k/n + y/n] = [m + z + y/n],
\]
and
\[
[[x]/n] = [k/n] = m,
\]
so it remains to show that \( z + y/n < 1 \). Note that \( k/n = m + z \) and \( n \geq 1 \) implies \( z = a/n \) for some \( a \) in \( 0, 1, \ldots, n - 1 \). Thus \( z \leq (n - 1)/n \). And since \( 0 \leq y < 1 \), we have \( y/n < 1/n \). Hence \( z + y/n < (n - 1)/n + 1/n = 1 \), and we are done.

3.14. If \( 0 < y < 1 \), what are the possible values of \([x] - [x - y]\)?

Proof. Let \( x = k + z \), where \( k \) is an integer and \( 0 \leq z < 1 \). If \( z \geq y \) then
\[
[x] - [x - y] = k - [k + z - y] = k - k = 0,
\]
and if \( z < y \) then
\[
[x] - [x - y] = k - [k + z - y] = k - [k - 1 + (1 - (y - z))] = k - (k - 1) = 1.
\]
Hence \([x] - [x - y]\) equals 0 or 1.

3.15. The number \( \{x\} = x - [x] \) is called the fractional part of \( x \). It satisfies the inequalities \( 0 \leq \{x\} < 1 \), with \( \{x\} = 0 \) if and only if \( x \) is an integer. What are the possible values of \( \{x\} + \{-x\}\)?

Proof. If \( x = [x] \) then the sum is trivially 0. Else, let \( x = k + y \), where \( k \) is an integer and \( 0 < y < 1 \). Then \( \{x\} = y \), and since \([-x] = -[x] - 1 \), we have \( \{-x\} = -x - ([-x] - 1) = -y + 1 \). Hence \( \{x\} + \{-x\} = y + (-y + 1) = 1 \).

3.16.

(a) Prove that \([2x] - 2[x]\) is either 0 or 1.

Proof. Let \( x = k + y \), where \( k \) is an integer and \( 0 \leq y < 1 \). If \( y < 1/2 \) then
\[
[2x] - 2[x] = 2k - 2k = 0,
\]
and if \( y \geq 1/2 \) then \( 2x = 2k + 2y \), where \( 1 \leq 2k < 2 \), so that
\[
[2x] - 2[x] = (2k + 1) - 2k = 1.
\]
(b) Prove that \([2x] + [2y] \geq [x] + [y] + [x + y]\).

**Proof.** Let \(x = m + a\) and \(y = n + b\), where \(m\) and \(n\) are integers and \(0 \leq a, b < 1\).

We consider 4 cases:

1. \(a, b < 1/2\),
2. \(a < 1/2, b \geq 1/2, a + b < 1\),
3. \(a < 1/2, b \geq 1/2, a + b \geq 1\),
4. \(a, b \geq 1/2\).

Combined with symmetry of \(a\) and \(b\) in cases (2) and (3), these cover all cases.

Case (1): \([2x] + [2y] = 2m + 2n = [x] + [y] + [x + y] = m + n + (m + n)\).

Case (2): \([2x] + [2y] = 2m + (2n + 1) > [x] + [y] + [x + y] = m + n + (m + n)\).

Case (3): \([2x] + [2y] = 2m + 1 + (2n + 1) > [x] + [y] + [x + y] = m + n + (m + n + 1)\).

This shows the statement holds in all cases.

**3.17.** Prove that \([x] + [x + \frac{1}{2}] = [2x]\) and, more generally,

\[\sum_{k=0}^{n-1} \left[ x + \frac{k}{n} \right] = \lfloor nx \rfloor.\]

**Proof.** Let \(x = m + a\) where \(m\) is an integer and \(0 \leq a < 1\). If \(a < \frac{1}{2}\) then

\([x] + [x + \frac{1}{2}] = m + m = 2m = [2x]\),

and if \(a \geq \frac{1}{2}\), then

\([x] + [x + \frac{1}{2}] = m + (m + 1) = 2m + 1 = [2x]\).

More generally, let \(j\) be the integer in \(0, 1, \ldots, n - 1\) such that \(j/n \leq a < (j + 1)/n\). Then

\[\sum_{k=0}^{n-1} \left[ x + \frac{k}{n} \right] = \sum_{k=0}^{n-j-1} \left[ x + \frac{k}{n} \right] + \sum_{k=n-j}^{n-1} \left[ x + \frac{k}{n} \right] = (n - j)m + j(m + 1) = nm + j = \lfloor nx \rfloor.\]
3.18. Let \( f(x) = x - [x] - \frac{1}{2} \). Prove that
\[
\sum_{k=0}^{n-1} f \left( x + \frac{k}{n} \right) = f(nx)
\]
and deduce that
\[
\left| \sum_{n=1}^{m} f \left( 2^n x + \frac{1}{2} \right) \right| \leq 1 \text{ for all } m \geq 1 \text{ and all } x.
\]

Proof. By using the previous exercise, 3.17, we have
\[
\sum_{k=0}^{n-1} f \left( x + \frac{k}{n} \right) = \sum_{k=0}^{n-1} \left( x + \frac{k}{n} - \left[ x + \frac{k}{n} \right] - \frac{1}{2} \right)
= nx + \frac{n-1}{2} - \lfloor nx \rfloor - \frac{n}{2}
= nx - \lfloor nx \rfloor - \frac{1}{2}
= f(nx).
\]

For the second part, note that
\[
\sum_{k=0}^{1} f \left( 2^n x + \frac{k}{2} \right) = f(2 \cdot 2^n x),
\]
so
\[
f(2^n x) + f \left( 2^n x + \frac{1}{2} \right) = f(2^{n+1} x).
\]

Thus for each \( n \), we have
\[
f \left( 2^n x + \frac{1}{2} \right) = f(2^{n+1} x) - f(2^n x),
\]
and summing from \( n = 1 \) to \( m \) results in
\[
\left| \sum_{n=1}^{m} f \left( 2^n x + \frac{1}{2} \right) \right| = \left| \sum_{n=1}^{m} (f(2^{n+1} x) - f(2^n x)) \right|
= \left| f(2^m x) - f(2x) \right|
\leq 1
\]
since \( |f(x)| \leq \frac{1}{2} \) for all \( x \).

3.19. Given positive odd integers \( h \) and \( k \), \((h,k) = 1\), let \( a = (k-1)/2\), \( b = (h-1)/2\).
(a) Prove that \( \sum_{r=1}^{a} [hr/k] + \sum_{r=1}^{b} [kr/h] = ab \).

Proof. Consider the line in the plane that passes through the coordinates \((0,0)\) and \((h,k)\). Since \(h\) and \(k\) are relatively prime, this line does not intersect any lattice points in the rectangle with corners \((1,1)\) and \((h,k)\) except for the point \((h,k)\) itself, call this rectangle \(R\). The number of even lattice points, i.e. the points \((m,n)\) where \(m\) and \(n\) are both even, inside \(R\) is precisely \(ab\). Note that the first sum counts the number of even lattice points in the rectangle above the line, and the second sum counts the number of even lattice points below the line. Since there are no even lattice points on the line, the two sums combine to \(ab\).

(b) Obtain a corresponding result if \((h,k) = d\).

Proof. Use the technique in part (a). There are precisely \((d-1)/2\) even lattice points in the rectangle which are double counted by the sums. Hence we have
\[
\sum_{r=1}^{a} \left\lfloor \frac{hr}{k} \right\rfloor + \sum_{r=1}^{b} \left\lfloor \frac{kr}{h} \right\rfloor - \frac{d-1}{2} = ab.
\]

3.20. If \(n\) is a positive integer prove that \(\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor\).

Proof. Let \(n = k^2 + a\), where \(k\) and \(a\) are integers satisfying \(k = \lfloor \sqrt{n} \rfloor\) and \(0 \leq a < 2k + 1\). We consider two cases: \(a < k\) and \(a \geq k\). In the first case, we have on the left side
\[
\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{k^2 + a + \sqrt{k^2 + a + 1}} \rfloor = 2k,
\]
as
\[
k^2 + a + 1 < k^2 + k + \frac{1}{4} < \left(k + \frac{1}{2}\right)^2,
\]
so that both \(\sqrt{n}\) and \(\sqrt{n+1}\) are less than \(k + \frac{1}{2}\). Hence the sum is less than \(2k + 1\) and the floor is \(2k\). On the other side, we have
\[
\lfloor \sqrt{4n+2} \rfloor = \lfloor \sqrt{4k^2 + 4a + 2} \rfloor = \left\lfloor 2\sqrt{k^2 + a + \frac{1}{2}} \right\rfloor = 2k,
\]
as
\[
k^2 + a + \frac{1}{2} < k^2 + k + \frac{1}{4} = \left(k + \frac{1}{2}\right)^2,
\]
so that \(\sqrt{k^2 + a + \frac{1}{2}} < k + \frac{1}{2}\) and hence \(2\sqrt{k^2 + a + \frac{1}{2}} < 2k + 1\).

Now for the second case, \(a \geq k\). On the left side we have
\[
\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{k^2 + a + \sqrt{k^2 + a + 1}} \rfloor = 2k + 1,
\]
as
\[ k^2 + a + 1 \geq k^2 + k + 1 > k^2 + k + \frac{1}{4} = \left( k + \frac{1}{2} \right)^2, \]
and \( k^2 + a \) is also greater than this quantity if \( a > k \); if \( a = k \) then
\[
\sqrt{\left( k^2 + a + \frac{1}{4} \right) - \frac{1}{4}} + \sqrt{\left( k^2 + a + \frac{1}{4} \right) + \frac{3}{4}} \geq 2k + 1.
\]
On the right side we have
\[
[\sqrt{4n + 2}] = [\sqrt{4k^2 + 4a + 2}] = \left[ 2 \sqrt{\sqrt{k^2 + a} + \frac{1}{2}} \right] = 2k + 1,
\]
as
\[ k^2 + a + \frac{1}{2} > k^2 + k + \frac{1}{4} = \left( k + \frac{1}{2} \right)^2, \]
and hence \( \sqrt{k^2 + a + \frac{1}{2}} > k + \frac{1}{2} \).

3.21. Determine all positive integers \( n \) such that \( [\sqrt{n}] \) divides \( n \).

\textit{Proof.} Let \( n = k^2 + a \), where \( k \) and \( a \) are integers satisfying \( k = \lfloor \sqrt{n} \rfloor \) and \( 0 \leq a < 2k + 1 \). Then \( k|n \) if and only if \( n = k^2 + mk \) where \( m = 0, 1, 2 \). The if direction is trivial. For the only if direction, we see that since \( k|(k^2 + a) \) and \( k|k^2 \), this implies \( k|a \), hence \( a = mk \). Now suppose \( m \geq 3 \), then \( n = k^2 + mk \geq k^2 + 3k \geq k^2 + 2k + 1 = (k + 1)^2 \), contradicting the choice of \( k \) for \( [\sqrt{n}] \). Hence \( m = 0, 1, 2 \).

3.22. If \( n \) is a positive integer, prove that
\[
\left[ \frac{8n + 13}{25} \right] - \left[ \frac{n - 12 - \left\lfloor \frac{n - 17}{25} \right\rfloor}{3} \right]
\]
is independent of \( n \).

\textit{Proof.} Note that if we increase \( n \) by 25, both terms increase by precisely 8, canceling the effect. Thus to show that this difference has the same value for all \( n \geq 1 \), it suffices to show it has the same value for \( n = 1, 2, \ldots, 25 \). This can be shown by exhaustion, resulting in 4 for all \( n \).

3.23. Prove that
\[
\sum_{n \leq x} \lambda(n) \left[ \frac{x}{n} \right] = [\sqrt{x}].
\]

\textit{Proof.} Let \( s(n) \) be the indicator function for squares, namely
\[
s(n) = \begin{cases} 
1 & \text{if } n \text{ is a square}, \\
0 & \text{otherwise}.
\end{cases}
\]
Note that
\[ [\sqrt{x}] = \sum_{n \leq x} s(n), \]
and by Theorem 3.10,
\[ \sum_{n \leq x} \lambda(n) \left[ \frac{x}{n} \right] = \sum_{n \leq x} \sum_{d|n} \lambda(d) = \sum_{n \leq x} s(n), \]
so the two sides are equal.

3.24. Prove that
\[ \sum_{n \leq x} \left[ \sqrt{\frac{x}{n}} \right] = \sum_{n \leq \sqrt{x}} \left[ \frac{x}{n^2} \right]. \]

Proof. Once again let \( s(n) \) be the indicator function for squares, i.e. \( s(n) = 1 \) if \( n \) is a square and \( s(n) = 0 \) otherwise. Note that
\[ [\sqrt{n}] = \sum_{n \leq x} s(n). \]
By Theorem 3.10, the left side is
\[ \sum_{n \leq x} \left[ \sqrt{\frac{x}{n}} \right] = \sum_{n \leq x} (s * u)(n). \]
Rewriting the right side and applying Theorem 3.10 gives
\[ \sum_{n \leq \sqrt{x}} \left[ \frac{x}{n^2} \right] = \sum_{n \leq x} s(n) \left[ \frac{x}{n^2} \right] = \sum_{n \leq x} (s * u)(n), \]
hence the two sides are equal.

3.25. Prove that
\[ \sum_{k=1}^{n} \left[ \frac{k}{2} \right] = \left[ \frac{n^2}{4} \right] \]
and that
\[ \sum_{k=1}^{n} \left[ \frac{k}{3} \right] = \left[ \frac{n(n-1)}{6} \right]. \]

Proof. For the first equation, let \( n = 2m + r \), where \( m = \lfloor n/2 \rfloor \) and \( r = 0, 1 \).
The left side is
\[ \sum_{k=1}^{n} \left[ \frac{k}{2} \right] = 2 \sum_{k=0}^{m-1} k + (r + 1)m = m(m - 1) + (r + 1)m = m^2 + rm, \]
and the right side is
\[
\left[ \frac{n^2}{4} \right] = \left[ \frac{4m^2 + 4rm + r^2}{4} \right] = \left[ m^2 + rm + \frac{r^2}{4} \right],
\]
Since \( r = 0,1, \), \([r^2/4] = 0\), so the equation holds.

For the second equation, let \( n = 3m + r \), where \( m = \lfloor n/3 \rfloor \) and \( r = 0, 1, 2 \). The left side is
\[
\sum_{k=1}^{n} \left[ \frac{k}{3} \right] = 3 \sum_{k=0}^{m-1} k + (r + 1)m = \frac{3m^2 - m + 2rm}{2},
\]
and the right side is
\[
\left[ \frac{n(n-1)}{6} \right] = \left[ \frac{9m^2 + 6mr - 3m + r^2 - r}{6} \right] = \left[ \frac{3m^2 + 2mr - m + \frac{r^2-r}{3}}{2} \right].
\]
Since \( r = 0, 1, 2, \), \([r^2 - r)/3] = 0\), and the equation holds.

3.26. If \( a = 1, 2, \ldots, 7 \) prove that there exists an integer \( b \) (depending on \( a \)) such that
\[
\sum_{k=1}^{n} \left[ \frac{k}{a} \right] = \left[ \frac{(2n + b)^2}{8a} \right].
\]

Proof. We show that \( b = 2 - a \) works for \( a = 1, 2, \ldots, 7 \). As before, let \( n = am + r \), where \( m = \lfloor n/a \rfloor \) and \( r = 0, 1, \ldots, a - 1 \). Then
\[
\sum_{k=1}^{n} \left[ \frac{k}{a} \right] = a \sum_{k=0}^{m-1} k + (r + 1)m
= am^2 - am + 2rm + 2m
= \frac{2}{2}
\]
\[
= \frac{4a^2m^2 - 4a^2m + 8arm + 8am}{8a}.
\]
Now let \( b = 2 - a \) and write
\[
\left[ \frac{(2n + b)^2}{8a} \right] = \left[ \frac{(2am + 2r + 2 - a)^2}{8a} \right],
\]
where the numerator expanded out is \( 4a^2m^2 - 4a^2m + 8arm + 8am + 4r^2 + 8r - 4ar + 4 - 4a + a^2 \). To show (1) and (2) have the same value, we need to show the numerator of (2) minus the numerator of (1), call this difference \( d \), satisfies
\( 0 \leq d < 8a \). Note that
\[
d = 4r^2 + 8r - 4ar + 4 - 4a + a^2 = (2r + 2 - a)^2 \geq 0.
\]
Now \( r \leq a - 1 \) and \( a < 8 \) implies
\[
(2r + 2 - a)^2 \leq (2(a - 1) + 2 - a)^2 = a^2 < 8a,
\]
so we are done.