Riemann's Explicit Formula

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This paper is a brief study of Bernhard Riemann's main result in analytic number theory: the article "Über die Anzahl der Primzahlen unter einer gegebenen Grösse" (1859), in which he derives an explicit formula for the prime counting function. Much of our paper works to make Riemann's intuitive statements more rigorous. In fact, to prove some of his ideas, we need to use theorems that were not invented until decades after his lifetime.

1 Introduction

The theory begins with Euler's product formula, which states that for $s > 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

where p ranges over all primes. The fomula can be shown by expanding each term in the product as

$$\frac{1}{1 - \frac{1}{p^s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots$$

and multiplying out all of them. This results in an infinite sum of terms in the form

$$\frac{1}{(p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m})^s}$$

where $p_1, ..., p_m$ are distinct primes and $k_1, ..., k_m$ are positive integers. Then one may use the fundamental theorem of arithmetic, which states that every integer has a unique prime factorization, to see that each of these terms is a $1/n^s$. When summed, these terms equal the left-hand side.

Riemann called this function $\zeta(s)$ and considered its behavior when s is a complex variable. It is not hard to see that it converges in the halfplane $\mathrm{Re}(s) > 1$. Let $s = \sigma + it$ where σ and t are real. Then for $\sigma > 1$,

$$n^s = n^\sigma n^{it} = n^\sigma e^{it \log n},$$

so that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} e^{-it \log n}.$$

But $|e^{-it \log n}| = 1$, so the sum converges absolutely in the halfplane $\mathrm{Re}(s) > 1$. Moreover, in $\mathrm{Re}(s) > 1$ there is uniform convergence, so ζ is holomorphic in this halfplane.

2 Properties of $\zeta(s)$

To obtain a formula for $\zeta(s)$ that works when s is outside the halfplane $\mathrm{Re}(s) > 1$, we shall extend ζ to a meromorphic function in $\mathbb{C}$, using the gamma and theta functions.

2.1 The Gamma Function

Our first object of study is the gamma function, defined as

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

for $s > 0$. When s is a positive integer, $\Gamma(s) = (s-1)!$. To see that it converges, one may break it up into

$$\Gamma(s) = \int_0^1 e^{-t} t^{s-1} dt + \int_1^\infty e^{-t} t^{s-1} dt$$

and observe that the second integral defines an entire function, while the first can be dealt with accordingly. Expand e^{-t} as a power series and integrate termwise, resulting in

$$\int_0^1 e^{-t} t^{s-1} dt = \sum_1^\infty \frac{(-1)^n t^{n+s}}{n!(n+s)} \Bigg|_0^1 = \sum_1^\infty \frac{(-1)^n}{n!(n+s)},$$

which defines a meromorphic function on $\mathbb{C}$, having poles at the negative integers with residue $(-1)^n/n!$ at $s = -n$. This is easy to verify, as the rapid growth of $n!$ in the denominator makes the series converge uniformly. Therefore, the relation

$$\Gamma(s) = \sum_1^\infty \frac{(-1)^n}{n!(n+s)} + \int_1^\infty e^{-t} t^{s-1} dt$$

defines a meromorphic function.

Before going on, we first write out a property of the gamma function that shall be useful later:

$$\begin{aligned}\Gamma(s+1) &= \lim_{N\to\infty} \frac{N!}{(s+1)(s+2)\cdots(s+N)}(N+1)^s \\ &= \prod_{n=1}^{\infty} \frac{n^{1-s}(n+1)^s}{s+n} \\ &= \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^s}{\left(1+\frac{s}{n}\right)}. \end{aligned} \tag{1}$$

The first line is due to Euler, and the second and third are reformulations of it.

2.2 The Theta Function

For our case, define the theta function for real $t > 0$ as

$$\vartheta(t) = \sum_{-\infty}^{\infty} e^{-\pi n^2 t}.$$

This satisfies the functional equation

$$\vartheta(t) = t^{-\frac{1}{2}} \vartheta\left(\frac{1}{t}\right)$$

which can be shown by application of the Poisson summation formula to $\vartheta(t)$.

The growth of $\vartheta(t)$ is bounded like

$$|\vartheta(t) - 1| \leq Ce^{-\pi t}.$$

This can be seen from the fact that

$$\sum_{1}^{\infty} e^{-\pi n^2 t} \leq \sum_{1}^{\infty} e^{-\pi n t} \leq Ce^{-\pi t}$$

for $t \geq 1$. The behavior of $\vartheta(t)$ near $t = 0$ is given by

$$\vartheta(t) \leq Ct^{-\frac{1}{2}}$$

which can be seen from the functional equation.

2.3 Analytic Continuation and Functional Equation

Now we are in a position to relate ζ, γ, and ϑ as follows. The proof is based on Stein and Shakarchi [S1]. Let $\mathrm{Re}(s) > 1$. If $n \geq 1$, then

$$\int_0^\infty e^{-\pi n^2 u} u^{(s/2)-1} du = \pi^{-s/2}\Gamma(s/2) n^{-s},$$

which can be seen immediately from the change of variable $u = t/(\pi n^2)$, making the integral

$$\int_0^\infty e^{-t} t^{(s/2)-1} dt \cdot (\pi n^2)^{-s/2}$$

equal to $\pi^{-s/2}\Gamma(s/2)n^{-s}$. Now, because

$$\frac{\vartheta(u)-1}{2} = \sum_{n=1}^\infty e^{-\pi n^2 u},$$

and because of the previously shown bounds on the growth and decay of ϑ, we may interchange the sum and integral. Then

$$\begin{aligned}
\frac{1}{2}\int_0^\infty u^{(s/2)-1}[\vartheta(u)-1]du &= \sum_{n=1}^\infty \int_0^\infty u^{(s/2)-1} e^{-\pi n^2 u} du \\
&= \pi^{-s/2}\Gamma(s/2)\sum_{n=1}^\infty n^{-s} \\
&= \pi^{-s/2}\Gamma(s/2)\zeta(s)
\end{aligned}$$

Now define

$$\psi(u) = \frac{\vartheta(u)-1}{2}.$$

The functional equation $\vartheta(u) = u^{-1/2}\vartheta(1/u)$ implies

$$\psi(u) = u^{-1/2}\psi(1/u) + \frac{1}{2u^{1/2}} - \frac{1}{2}.$$

From the previously derived equation, we have, for $\mathrm{Re}(s) > 1$,

$$\begin{aligned}
\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \int_0^\infty u^{(s/2)-1}\psi(u)du \\
&= \int_0^1 u^{(s/2)-1}\psi(u)du + \int_1^\infty u^{(s/2)-1}\psi(u)du \\
&= \int_0^1 u^{(s/2)-1}\left[u^{-1/2}\psi(1/u) + \frac{1}{2u^{1/2}} - \frac{1}{2}\right] du \\
&\quad + \int_1^\infty u^{(s/2)-1}\psi(u)du \\
&= \frac{1}{s-1} + \frac{1}{s} + \int_1^\infty [u^{-(s/2)-1/2} + u^{(s/2)-1}]\psi(u)du.
\end{aligned}$$

Note that this defines a meromorphic function with simple poles at 0 and 1. This is because the exponential decay of ψ in the integral means the integral defines an entire function. Also, observe that the value is unchanged if s is replaced by $1-s$. Hence

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s),$$

which allows us to define values for zeta everywhere except the pole at $s = 1$.

We shall follow Riemann's notation and multiply $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ by the factor $s(s-1)/2$ and define this as*

$$\xi(s) = \Gamma(s/2+1)(s-1)\pi^{-s/2}\zeta(s).$$

The advantage of using this definition of ξ is that the multiplying by s and $s-1$ effectively cancel the simple poles of $\pi^{-s/2}\Gamma(s/2)\zeta(s)$, and hence $\xi(s)$ is an entire function and satisfies

$$\xi(s) = \xi(1-s).$$

We may rearrange to find

$$\zeta(s) = \frac{\pi^s \xi(s)}{(s-1)\Gamma(s/2+1)},$$

which shows that ζ has a simple pole at 1 and zeros where $\Gamma(s/2+1)$ has poles, namely at $s/2 = -n$, $n \in \mathbb{N}$. Hence zeta has simple zeros at -2, -4, -6, etc. These are defined the trivial zeros. Note that all other zeros of ζ must also be zeros of ξ. These nontrivial zeros are denoted by ρ.

Furthermore, the zeta function can be defined in the halfplane $\mathrm{Re}(s) > 1$ by

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}.$$

Now $(1-1/p^s)^{-1} - 1$ converges absolutely, so if zeta has a zero in this halfplane, then one of the terms $(1-1/p^s)^{-1}$ must equal zero. This is impossible, so ζ has no zeroes in the halfplane $\mathrm{Re}(s) > 1$. And from the functional equation $\xi(s) = \xi(1-s)$, it follows that there are no zeros in the halfplane $\mathrm{Re}(s) < 0$, except at the trivial zeros. Thus all of the nontrivial zeros must lie in the rectangle $0 \leq \mathrm{Re}(s) \leq 1$. This bound can be improved to remove the lines $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ and thus have the statement that all nontrivial zeros of zeta lie in the region $0 < \mathrm{Re}(s) < 1$, which is denoted as the critical strip.

2.4 Product Formula for $\xi(s)$

Riemann assumed it was possible to factor $\xi(s)$ in terms of its roots in something of the form

$$\xi(s) = f(s) \prod_\rho \left(1 - \frac{s}{\rho}\right),$$

where f(s) is a function that does not vanish. Given this was possible, he showed that $f(s)$ must be a constant, and then showed that the constant must be $f(s) = \xi(0)$, which follows upon setting $s = 0$.

*The ξ function is usually defined as $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, which has been shown to have simple poles at 0 and 1.

The factoring step is indeed valid as shown by Hadamard in 1893, some 34 years after the publication of Riemann's paper. We will not repeat the proof of the Hadamard factorization here, as it is a fairly intricate result (a proof can be found starting on p.147 of [S1]). The factorization theorem states for this case that $f(s) = e^{a+bs}$ because ξ has order of growth 1 (this can be easily checked from the equation defining ξ). Then since $\xi(s-1/2)$ is an even function (this follows from $\xi(s) = \xi(1-s)$), $\operatorname{Re}\log\xi(s-1/2)$ is an even function but must grow slower than $s^{1+\epsilon}$. A linear term cannot be even, so it must be constant. Hence we have the equation

$$\xi(s) = \xi(0)\prod_{\rho}\left(1 - \frac{s}{\rho}\right).$$

But we also have by definition that

$$\xi(s) = \Gamma(s/2+1)(s-1)\pi^{-s/2}\zeta(s),$$

so we may combine these, take the log, and rearrange to obtain

$$\log\zeta(s) = \log\xi(0) + \sum_{\rho}\log\left(1-\frac{s}{\rho}\right) - \log\Gamma\left(\frac{s}{2}+1\right) + \frac{s}{2}\log\pi - \log(s-1). \quad (2)$$

3 Building the Formula

3.1 $\pi(x)$ and J(x)

The end goal is to obtain a formula for $\pi(x)$, which counts the number of primes less than x. For our purposes, we shall use the formula

$$\pi(x) = \frac{1}{2}\left[\sum_{p<x} 1 + \sum_{p\le x} 1\right].$$

This function starts at 0 when $x = 0$ and jumps by 1 at each prime. At each jump, the function assumes the halfway value. Since $\pi(x)$ almost everywhere assumes integer values, it is difficult to imagine why a formula based on analytic techniques should exist.

Riemann next defined the function $J(x)$. Like $\pi(x)$, this function starts at 0 when $x = 0$ and jumps by 1 for every prime, but it also jumpts by 1/2 for every prime square, 1/3 for every prime cube, etc. It may be defined as

$$J(x) = \frac{1}{2}\left[\sum_{p^n<x}\frac{1}{n} + \sum_{p^n\le x}\frac{1}{n}\right]$$

where it assumes halfway values at the jumps. The reason this function is interesting is that it may be related to the zeta function as follows.

Consider the product formula of $\zeta(s)$ for $\mathrm{Re}(s) > 1$

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}.$$

Taking the log of both sides and using the Taylor series for the log yields

$$\begin{aligned}
\log \zeta(s) &= \sum_p -\log\left(1 - \frac{1}{p^s}\right) \\
&= \sum_p \left(\frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \cdots\right) \\
&= \sum_p \sum_n \frac{p^{-ns}}{n}.
\end{aligned}$$

Observe that

$$p^{-ns} = s \int_{p^n}^{\infty} x^{-s-1} dx$$

which follows from elementary calculus. We may substitute this into the $\log \zeta(s)$ formula to obtain

$$\log \zeta(s) = s \sum_p \sum_n \frac{1}{n} \int_{p^n}^{\infty} x^{-s-1} dx$$

Because this is absolutely convergent for $\mathrm{Re}(s) > 1$, it follows that we may interchange the order of summation and integration, resulting in[†]

$$\begin{aligned}
\log \zeta(s) &= s \int_0^{\infty} \sum_{p^n < x} \frac{1}{n} x^{-s-1} dx \\
&= s \int_0^{\infty} J(x) x^{-s-1} dx. \qquad (3)
\end{aligned}$$

This is the key relation between $\zeta(s)$ and $J(x)$. Later in this section we shall use this formula again.

Now we need a relation between $J(x)$ and $\pi(x)$. This is given by

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots \qquad (4)$$

where the number of primes less than x is counted with weight 1, the number of prime squares less than x is counted with weight 1/2, etc. Note that the sum is actually a finite sum, as $\pi(x) = 0$ for $x < 2$ (there are no primes less than 2). This shall be helpful, though not necessary, for inverting the relation.

[†]Note that since jumps occur on a set of measure zero, it does not matter in the sum whether we use $p^n < x$ or $p^n \leq x$.

The method of inversion will be the Möbius inversion. Let $\mu(n)$ denote the Möbius function, defined for $n \in \mathbb{N}$ as

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^k, & \text{if } n \text{ is the product of } k \text{ distinct primes,} \\ 0, & \text{otherwise.} \end{cases}$$

Then Möbius inversion on equation (4) gives

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(x^{\frac{1}{n}})$$

which is also a finite sum, for when $x < 2$, we have $J(x) = 0$ (there are no primes or prime powers less than 2). So, all the terms where $x^{\frac{1}{n}} < 2$ are 0, that is, which means there are only $\lfloor \log x / \log 2 \rfloor$ non-zero terms.

At this point, note that since $J(x)$ counts primes and weighted prime powers below x, $J(x)$ grows no faster than x (in fact, the prime number theorem implies $J(x) \sim x/\log x$). Then the function $J(x)x^{-s-1}$ grows slower than x^{-s}. Combine this with the fact that $J(x) = 0$ for $x < 2$, to see that $J(x)x^{-s-1}$ is integrable across the line when $\mathrm{Re}(s) > 1$. So, we may use the inverse Laplace transform on the equation

$$\frac{\log \zeta(s)}{s} = \int_0^{\infty} J(x) x^{-s-1} dx$$

which is a reassembling of equation (3) to find

$$J(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s) \frac{x^s}{s} ds \tag{5}$$

with $a > 1$.

3.2 The Product Formula and the Result

The next step begins a long line of hard work. We now attempt to substitute equation (2), reprinted below,

$$\log \zeta(s) = \log \xi(0) + \sum_{\rho} \log \left(1 - \frac{s}{\rho}\right) - \log \Gamma \left(\frac{s}{2} + 1\right) + \frac{s}{2} \log \pi - \log(s-1)$$

into (5). If this works, then we can integrate term-wise and obtain a formula for $J(x)$. Unfortunately, the direct substitution does not work because it leads to divergent integrals. We can, however, first integrate (5) by parts to obtain

$$J(x) = -\frac{1}{2\pi i} \cdot \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{\log \zeta(s)}{s}\right] x^s ds \tag{6}$$

and then carry out the processes of substitution and term-wise integegration to obtain the desired formula. The integration by parts of (5) depends on the behavior of the term

$$\frac{1}{2\pi i} \cdot \frac{1}{\log x} \cdot \frac{\log \zeta(s)}{s} x^s$$

when $s \to a \pm \infty$. To prove the validity of (6), it suffices to show that

$$\lim_{T\to\infty} \frac{\log \zeta(a \pm iT)}{a \pm iT} x^{a \pm iT} = 0.$$

This follows from the inequality

$$|\log \zeta(a \pm iT)| = \left| \sum_n \sum_p (1/n) p^{-n(a\pm iT)} \right| \leq \sum_n \sum_p (1/n) p^{-na} = \log \zeta(s) < \infty$$

so that the numerator is bounded, the denominator goes to infinity, and the right-hand term is also bounded. Hence the term goes to zero and the integration by parts is valid. The next section, in which we integrate term-wise, is the hard part.

4 The Terms of J(x)

After substitution, formula (6) at the end of the last section gives us an integral with 5 terms. The evaluations of some of these integrals are certainly not trivial. Much of the work in this section is due to Edwards [E1].

For sake of nearby reference for the reader, the integral is

$$J(x) = -\frac{1}{2\pi i} \cdot \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{\log \zeta(s)}{s} \right] x^s ds$$

and the terms are

$$\log \zeta(s) = \log \xi(0) + \sum_\rho \log \left(1 - \frac{s}{\rho}\right) - \log \Gamma\left(\frac{s}{2} + 1\right) + \frac{s}{2} \log \pi - \log(s-1),$$

derived in the previous sections.

4.1 The Main Term

We shall start with the $-\log(s-1)$ term. This becomes

$$\frac{1}{2\pi i} \cdot \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{\log(s-1)}{s} \right] x^s ds.$$

To compute this integral, we first define a few auxiliary functions, the the first of which is

$$F(\beta) = \frac{1}{2\pi i} \cdot \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left\{ \frac{\log[(s/\beta) - 1]}{s} \right\} x^s ds$$

where our term in question is the special case $F(1)$. To extend F, we take $a > \operatorname{Re}\beta$ and define $\log[(s/\beta) - 1]$ as $\log(s-\beta)$ - $\log\beta$, to follow the principal branch of log. Moreover, the integral is absolutely convergent because

$$\left|\frac{d}{ds}\frac{\log[(s/\beta)-1]}{s}\right| \leq \frac{|\log[(s/\beta)-1]|}{|s|^2} + \frac{1}{|s(s-\beta)|}$$

is integrable, while x^s oscillates on the line of integration. Now we use the derivative

$$\frac{d}{d\beta}\frac{\log[(s/\beta)-1]}{s} = \frac{1}{(\beta-s)\beta}$$

to obtain

$$\begin{aligned} F'(\beta) &= \frac{1}{2\pi i}\cdot\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{1}{(\beta-s)\beta}\right]x^s ds \\ &= -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{x^s}{(\beta-s)\beta}ds \\ &= \frac{1}{2\pi i\beta}\int_{a-i\infty}^{a+i\infty}\frac{x^s}{s-\beta}ds \end{aligned}$$

where the first step comes from differentiation under the integral sign, the second from integration by parts, and the third from trivial rearrangement.

This can be computed. Consider the function

$$\frac{1}{s-\beta} = \int_1^\infty x^{-s}x^{\beta-1}dx \qquad [\operatorname{Re}(s-\beta) > 0].$$

Substitute $x = e^\lambda, dx = e^\lambda d\lambda$ and write $s = a + i\mu$ to obtain

$$\frac{1}{a+i\mu-\beta} = \int_0^\infty e^{-i\lambda\mu}e^{\lambda(\beta-a)}d\lambda \qquad [a > \operatorname{Re}(\beta)],$$

which gives, from Fourier inversion,

$$\int_{-\infty}^\infty \frac{1}{a+i\mu-\beta}e^{i\mu x}d\mu = \begin{cases} 2\pi e^{x(\beta-a)}, & \text{if } x > 0, \\ 0, & \text{if } x < 0. \end{cases}$$

It follows that

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{s-\beta}y^s ds = \begin{cases} y^\beta, & \text{if } y > 1, \\ 0, & \text{if } y < 1. \end{cases} \tag{7}$$

Since we already have $x > 1$, $F'(\beta) = x^\beta/\beta$.

The next step is to evaluate a contour integral. Let C^+ be the contour from 0 to x that consists of the real line segment from 0 to $1-\epsilon$, the semicircle in the upper halfplane $\operatorname{Im} t \geq 0$ from $1-\epsilon$ to $1+\epsilon$, and then the real line segment from $1+\epsilon$ to x. Define

$$G(\beta) = \int_{C^+}\frac{t^{\beta-1}}{\log t}dt$$

and note that

$$G'(\beta) = \int_{C+} t^{\beta-1} dt = \left.\frac{t^\beta}{\beta}\right|_0^x = F'(\beta).$$

Since $G(\beta)$ is defined and analytic for $\mathrm{Re}(\beta) > 0$, $G(\beta)$ and $F(\beta)$ must differ by a constant. The hope is that we can compute this constant and hence find $F(\beta)$ as $G(\beta)$ plus a constant.

We shall evaluate the constant by setting $\beta = \sigma + i\tau$, holding σ fixed, letting $\tau \to \infty$, and evaluate $F(\beta)$ and $G(\beta)$. First, we evaluate the limit of $G(\beta)$. Making the change of variable $t = e^u$ puts $G(\beta)$ in the form

$$\int_{i\delta-\infty}^{i\delta+\log x} \frac{e^{\beta u}}{u} du + \int_{i\delta+\log x}^{\log x} \frac{e^{\beta u}}{u} du.$$

Note that the path of integration has been altered slightly based on Cauchy's integral theorem. The further changes of variable $u = i\delta + v$ in the first integral and $u = \log x + iw$ in the second put $G(\beta)$ in the form

$$e^{i\delta\sigma} e^{-\sigma\tau} \int_{-\infty}^{\log x} \frac{e^{\sigma v}}{i\delta + v} e^{i\tau v} dv - i x^\beta \int_0^\delta \frac{e^{-\tau w} e^{\sigma i w}}{\log x + iw} dw,$$

whose values both approach 0 as $\tau \to \infty$. In the first integral, $e^{-\delta\tau} \to 0$ is enough to make the value 0, and in the second, $e^{-\tau w} \to 0$ except at $w = 0$. Therefore, the limit of $G(\beta)$ as $\tau \to \infty$ is 0.

Evaluating the limit of $F(\beta)$ is a bit trickier. Define another auxiliary function

$$H(\beta) = \frac{1}{2\pi i} \cdot \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left\{ \frac{\log[1 - (s/\beta)]}{s} \right\} x^s ds$$

where $a > \mathrm{Re}\,\beta$ and $\log[1-(s/\beta)]$ is defined for complex β as $\log(s-\beta) - \log(-\beta)$. The goal is to compare this to $F(\beta)$ and thereby to $G(\beta)$. In the upper halfplane $\mathrm{Im}\,\beta > 0$, the difference

$$\begin{aligned}
H(\beta) - F(\beta) &= \frac{1}{2\pi i} \cdot \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{\log \beta - \log(-\beta)}{s} \right] x^s ds \\
&= \frac{1}{2\pi i} \cdot \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{i\pi}{s} \right] x^s ds \\
&= -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{i\pi}{s} x^s ds \\
&= -i\pi
\end{aligned}$$

where the last result is derived from equation (7). Therefore, $F(\beta) = H(\beta) + i\pi$ in the upper halfplane, reducing the problem to finding the limit of $H(\beta)$ as $\tau \to \infty$. From the derivative

$$\begin{aligned}
\frac{d}{ds} \frac{\log[1 - (s/\beta)]}{s} &= -\frac{\log[1 - (s/\beta)]}{s^2} + \frac{1}{s(s-\beta)} \\
&= -\frac{\log[1 - (s/\beta)]}{s^2} + \frac{1}{\beta(s-\beta)} - \frac{1}{\beta s},
\end{aligned}$$

we may put this in the integral defining $H(\beta)$. The first term is then

$$-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\log[1-(s/\beta)]}{s^2}x^s ds.$$

Since $1-(s/\beta)\to 1$ and hence $\log[1-(s/\beta)]\to 0$, the numerator is strongly bounded. The denominator is s^2, which grows like $|s|^2$ for large τ, and x^s oscillates along the line of integration. The $1/s^2$ growth rate means the we may use the Lebesgue bounded convergence theorem so that the limit of the integral is the integral of the limit, which is 0 due to $\log[1-(s/\beta)]$ in the numerator. Hence, this integral is 0. The second and third terms combine to give

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left[\frac{1}{\beta(s-\beta)}-\frac{1}{\beta s}\right]x^s ds=\frac{x^\beta}{\beta}-\frac{1}{\beta}$$

from equation (7). The numerators are bounded and $|\beta|\to\infty$, hence these terms go to 0, and the function $H(\beta)$ goes to 0. This implies $F(\beta)\to i\pi$, and thus $F(\beta)=G(\beta)+i\pi$ in the halfplane $\operatorname{Re}\beta>0$. Finally, this allows us to write the main $J(x)$ term as

$$F(1)=\int_0^{1-\epsilon}\frac{dt}{\log t}+\int_{1-\epsilon}^{1+\epsilon}\frac{dt}{\log t}+\int_{1+\epsilon}^{x}\frac{dt}{\log t}+i\pi$$

Taking the limit as $\epsilon\to 0$, we see that the second term approaches along a pole of residue 1, but the contour is taken with the negative orientation, resulting in $\int_{1-\epsilon}^{1+\epsilon}\frac{dt}{\log t}=-i\pi$, from the residue theorem. This implies that the $i\pi$ terms cancel and we are left with

$$F(1)=\lim_{\epsilon\to 0}\int_0^{1-\epsilon}\frac{dt}{\log t}+\int_{1+\epsilon}^{x}\frac{dt}{\log t}=Li(x).$$

4.2 The Oscillatory Term

Next, we shall look at the term

$$\sum_\rho \log\left(1-\frac{s}{\rho}\right)$$

which involves the nontrivial roots of the zeta function. In the integral form for $J(x)$, this becomes

$$-\frac{1}{2\pi i}\cdot\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\sum_\rho \log\left(1-\frac{s}{\rho}\right)}{s}\right]x^s ds. \tag{8}$$

At this point it is not clear what to do, since we do not know whether the integral and sum can be interchanged. Riemann did not know how to prove this, but he assumed it could be done. We will see in a later section that if we assume the interchange is valid, the final result is the correct one, despite the possible invalidity of the method.

Assuming we can interchange the integral and sum, this expression becomes

$$-\sum_{\rho} H(\rho)$$

with the same $H(\rho)$ as defined in the previous section. We showed that $H(\rho) = G(\rho)$ in the first quadrant ($\operatorname{Re}\rho > 0, \operatorname{Im}\rho > 0$), and if we take the integral defining $G(\rho)$ to go through the lower halfplane, the same holds ρ in the fourth quadrant ($\operatorname{Re}\rho > 0, \operatorname{Im}\rho \leq 0$). That is, let C^- be the contour that goes in a line segment from 0 to $1-\epsilon$, in a semicircle in the lower halfplane ($\operatorname{Im}\rho < 0$) from $1-\epsilon$ to $1+\epsilon$, and then in a line segment from $1+\epsilon$ to x. Then after pairing the terms ρ and $1-\rho$, we find that the total sum is equal to

$$-\sum_{\operatorname{Im}\rho>0} \left(\int_{C^+} \frac{t^{\rho-1}}{\log t} dt + \int_{C^-} \frac{t^{-\rho}}{\log t} dt \right).$$

If β is real and positive, then the change of variable $u = t^\beta$, $\log t = (\log u)/\beta$, $dt/t = du/u\beta$ gives

$$\int_{C^+} \frac{t^{\beta-1}}{\log t} dt = \int_0^{x^\beta} \frac{du}{\log u} = Li(x^\rho) - i\pi,$$

where the path from 0 to x^β passes in the upper halfplane near $u = 1$. Now the integral converges in the upper halfplane $\operatorname{Re}\beta > 0$ and thus gives an analytic continuation of $Li(x^\beta)$ in this halfplane. On the other hand, the integral

$$\int_{C^-} \frac{t^{\beta-1}}{\log t} dt = Li(x^\rho) + i\pi,$$

through a similar argument. Thus the formula for equation (8) is

$$-\sum_{\operatorname{Im}\rho>0} \left[Li(x^\rho) + Li(x^{1-\rho}) \right].$$

We must be careful as this sum converges only conditionally. We take the sum in order of increasing $|\operatorname{Im}(\rho)|$.

4.3 The Constant Term

The next term is

$$\log \xi(0),$$

which becomes, in the integral,

$$-\frac{1}{2\pi i} \cdot \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{\log \xi(0)}{s} \right] x^s ds.$$

Integrating by parts and using equation (7), we have that the above is equal to

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\log\xi(0)}{s}x^s ds = \log\xi(0),$$

which is given by $\xi(0) = \Gamma(1)\pi^0(0-1)\zeta(0) = -\zeta(0) = \frac{1}{2}$, so that

$$\log\xi(0) = -\log 2.$$

4.4 The Integral Term

The last useful term is

$$\log\Gamma\left(\frac{s}{2}+1\right)$$

and the corresponding integral is

$$\frac{1}{2\pi i}\cdot\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log\Gamma\left(\frac{s}{2}+1\right)}{s}\right]x^s ds \tag{9}$$

Using formula (1), a property of the gamma function, we may rewrite

$$\log\Gamma\left(\frac{s}{2}+1\right) = \sum_1^\infty\left[-\log\left(1+\frac{s}{2n}\right)+\frac{s}{2}\log\left(1+\frac{1}{n}\right)\right].$$

Putting this formula in (9) and and assuming that we can interchange the sum and integral, we have (9) in the form

$$-\frac{1}{2\pi i}\cdot\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left\{\frac{-\log[1+(s/2n)]}{s}\right\}x^s ds$$

where only the first sum is intact (the second sum vanishes because division by s results in a constant, which has derivative 0). But this is equal to

$$-\sum_1^\infty H(-2n)$$

where H is defined as in section 4.1 in the evaluation of the main term. In that section we only evaluated H for $\mathrm{Re}(\beta) > 0$. To analyze the behavior of H in $\mathrm{Re}(\beta) < 0$, define

$$E(\beta) = -\int_x^\infty\frac{t^{\beta-1}}{\log t}dt$$

and note that

$$E'(\beta) = -\int_x^\infty t^{\beta-1}dt = \frac{x^\beta}{\beta} = F'(\beta) = H'(\beta),$$

so that $E(\beta)$ and $H(\beta)$ differ by a constant. Now both E and H approach zero as $\beta \to \infty$ and so the constant is zero, giving $E(\beta) = H(\beta)$. Thus our term becomes

$$\begin{aligned} -\sum_1^\infty H(-2n) &= \int_x^\infty \frac{t^{-2n-1}}{\log t} dt \\ &= \int_x^\infty \frac{1}{t \log t} \sum_1^\infty \left(t^{-2n}\right) dt \\ &= \int_x^\infty \frac{dt}{t(t^2-1)\log t} \end{aligned}$$

assuming that termwise integration is valid.

To show that it is, we consider

$$\frac{d}{ds} \frac{\log \Gamma(s/2+1)}{s} = -\sum_1^\infty \frac{d}{ds} \frac{-\log[1+(s/2n)]}{s}.$$

For large n, take the Taylor series expansion $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$ to find that

$$-\sum_1^\infty \frac{d}{ds} \frac{-\log[1+(s/2n)]}{s} = \frac{1}{2}\frac{1}{4n^2} - \frac{2}{3}\frac{s}{8n^3} + \frac{3}{4}\frac{s^2}{16n^4} - \cdots$$

which converges uniformly as the highest order term of n is n^{-2}. This justifies termwise differentiation. The termwise integration is likewise justified, as the terms decay like $1/n^2$ and the sum is hence uniformly convergent.

4.5 The Vanishing Term

The final term we look at is

$$\frac{s}{2} \log \pi,$$

which, as it turns out, completely vanishes in the formula for $J(x)$, because

$$-\frac{1}{2\pi i} \cdot \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{\frac{s}{2}\log \pi}{s} \right] x^s ds = 0.$$

The term is divided by s and becomes constant, resulting in a derivative of 0, and thus the entire term is 0.

4.6 Result

In the final analysis, we have

$$J(x) = Li(x) + \sum_\rho Li(x^\rho) - \log 2 + \int_x^\infty \frac{dt}{t(t^2-1)\log t}$$

with $x > 1$, and with the sum in the second term only conditionally convergent (one must sum in order of increasing $|\operatorname{Im}(\rho)|$). Combining this formula with

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(x^{\frac{1}{n}})$$

gives an analytic formula for $\pi(x)$. Remembering that this formula involves a finite sum, we can see easily that if the formula for $J(x)$ is valid, then the formula for $\pi(x)$ must also be valid.

We have not yet shown the validity of termwise integration for the second term

$$\sum_{\rho} Li(x^{\rho}).$$

A proof dealing with this sum directly was not discovered until 1908, nearly half a century after Riemann's paper, by Landau [L1]. There were also methods of indirect proof which involved formulas for functions similar to $J(x)$, one of which we shall examine in the next section.

5 The Von Mangoldt Formula

5.1 Deriving the Formula

Consider a counting function that counts primes and prime powers weighted by the log of the prime, that is,

$$\psi(x) = \sum_{p^n < x} \log p$$

where the function assumes the halfway value at each jump.

This function has the corresponding equation (proved by von Mangoldt in 1894, see [E1])

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi) + \sum_{n} \frac{x^{-2n}}{2n}$$

for $x > 1$. While we shall not fully prove it here, we can show that it is a very reasonable result. One can differentiate the formula for $J(x)$ to obtain

$$dJ = \left(\frac{1}{\log x} - \sum_{\rho} \frac{x^{\rho-1}}{\log x} - \frac{1}{x(x^2-1)\log x} \right) dx.$$

Now, since J jumps by $1/n$ at prime powers, $dJ = 1/n$ at $x = p^n$. Similarly, $d\psi = \log p = (1/n)\log(p^n) = (1/n)\log(x)$ at $x = p^n$. They are 0 everywhere else. Hence these equations give

$$\begin{aligned} d\psi &= (\log x) dJ \\ &= \left(1 - \sum_{\rho} x^{\rho-1} - \sum_{n} x^{-2n-1} \right) dx \end{aligned}$$

where the last term can be derived with geometric series. This leads to the plausible guess that

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} + \sum_{n} \frac{x^{-2n}}{2n} + C.$$

The hard part in showing that von Mangoldt's formula holds is showing that the oscillatory term , i.e. $\sum_\rho x^\rho/\rho$, converges.

To derive such a formula for $\psi(x)$ in terms of ζ, von Mangoldt used the same method as Riemann, i.e., he first found a formula for $\zeta(s)$ in an integral form of $\psi(x)$, and then took the Laplace transform. In his case, he found

$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_0^\infty \psi(x) x^{-s-1} dx,$$

which comes from log-differentiating the product formula for zeta, and then he applies the transform to obtain

$$\psi(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left[-\frac{\zeta'(s)}{\zeta(s)} \right] x^s \frac{ds}{s} \tag{10}$$

for $a > 1$.

For the next step, we shall find a formula for $-\zeta'(s)/\zeta(s)$ and take the integral termwise. The reader will probably recognize this process as nearly identical so far to the process Riemann used to find $J(x)$.

Using the equation

$$\xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) = \Gamma(s/2+1)(s-1)\pi^{-s/2}\zeta(s)$$

developed at the end of section 2.4 and log-differentiating, we find that

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \sum_{\rho} \frac{1}{s-\rho} + \sum_{n} \left[-\frac{1}{s+2n} + \frac{1}{2} \log\left(1 + \frac{1}{n}\right) \right] - \frac{1}{2} \log \pi.$$

Plugging $s = 0$ gives

$$-\frac{\zeta'(0)}{\zeta(0)} = -1 + \sum_{\rho} \frac{1}{\rho} + \sum_{n} \left[-\frac{1}{2n} + \frac{1}{2} \log\left(1 + \frac{1}{n}\right) \right] - \frac{1}{2} \log \pi,$$

which, when substracted from the previous equation, gives

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{s}{s-1} - \sum_{\rho} \frac{s}{\rho(s-\rho)} + \sum_{n} \frac{s}{2n(s+2n)} - \frac{\zeta'(0)}{\zeta(0)}. \tag{11}$$

5.2 The $\sum_\rho \frac{x^\rho}{\rho}$ Term

When we plug equation (11) into the integral in equation (10), the terms actually converge so we do not need to take the extra step of integrating by parts as we did for $J(x)$. This simplifies the calculation immensely. We shall skip over the calculation of the first, third, and fourth terms, as we already know from calculation of $J(x)$ what they should be (except for the value of the constant) and why they converge. We shall concern ourselves with the second term, arising from the nontrivial zeros, namely

$$-\sum_\rho \frac{s}{\rho(s-\rho)}$$

with the integral expression

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left[\sum_\rho \frac{s}{\rho(s-\rho)}\right] x^s \frac{ds}{s}. \tag{12}$$

The goal will be to show that this term converges and is equal to

$$\sum_\rho \frac{x^\rho}{\rho}. \tag{13}$$

If we pair the roots ρ and $1-\rho$ (such roots exist because $\xi(s)=\xi(1-s)$), we find that the sum actually converges uniformly. This can be seen from

$$\begin{aligned}\left|\frac{1}{s-\rho}+\frac{1}{s-(1-\rho)}\right| &= \left|\frac{1}{(s-\frac{1}{2})-(\rho-\frac{1}{2})}+\frac{1}{(s-\frac{1}{2})+(\rho-\frac{1}{2})}\right| \\ &= \left|\frac{2(s-\frac{1}{2})}{(s-\frac{1}{2})^2-(\rho-\frac{1}{2})^2}\right| \\ &\leq C\left|\frac{1}{(\rho-\frac{1}{2})^2}\right|\end{aligned}$$

for large ρ, and the fact that

$$\sum_\rho \frac{1}{\left|\rho-\frac{1}{2}\right|^{1+\epsilon}} < \infty,$$

which is essentially due to $\xi(s)$ having order of growth 1. The uniform convergence implies that this sum can be integrated termwise over finite intervals. Thus the term (12) is equal to

$$\lim_{h\to\infty}\sum_\rho \frac{1}{2\pi i}\int_{a-ih}^{a+ih}\frac{x^s ds}{\rho(s-\rho)} = \lim_{h\to\infty}\sum_\rho \frac{x^\rho}{\rho}\frac{1}{2\pi i}\int_{a-ih}^{a+ih}\frac{x^{s-\rho}ds}{s-\rho}$$

and defines the correct term in the formula for $\psi(x)$. It is not hard to find, for $x>1$,

$$\lim_{h\to\infty}\frac{1}{2\pi i}\int_{a-ih}^{a+ih}\frac{x^{s-\rho}ds}{s-\rho} = 1,$$

which follows immediately from the formula

$$\lim_{h\to\infty} \frac{1}{2\pi i}\int_{a-ih}^{a+ih} \frac{y^s ds}{s} = \begin{cases} 0, & \text{if } 0 < y < 1, \\ \frac{1}{2}, & \text{if } y = 1, \\ 1, & \text{if } y > 1. \end{cases}$$

This would imply that the term (12) converges to

$$\lim_{h\to\infty} \sum_\rho \frac{x^\rho}{\rho}\frac{1}{2\pi i}\int_{a-ih}^{a+ih} \frac{x^{s-\rho} ds}{s-\rho} = \sum_\rho \frac{x^\rho}{\rho}$$

if we are allowed to interchange the limit and sum. If this is possible, then we will have shown that (13) converges.

To do this, we shall follow von Mangoldt's proof, which takes the limit "diagonally" using the function

$$\sum_{|\operatorname{Im}(\rho)|\le h} \frac{x^\rho}{\rho}\frac{1}{2\pi i}\int_{a-ih}^{a+ih} \frac{x^{s-\rho} ds}{s-\rho}. \tag{14}$$

Before doing the proof, we need two bounds on the integral

$$\frac{1}{2\pi i}\int_{a-ih}^{a+ih} \frac{y^s ds}{s}.$$

The first bound is

$$\left| \frac{1}{2\pi i}\int_{a-ih}^{a+ih} \frac{y^s ds}{s} - 1 \right| \le \frac{x^a}{\pi h \log x} \tag{15}$$

with $x > 1$ and $a > 0$, and the second is

$$\left| \frac{1}{2\pi i}\int_{a-ic}^{a+id} \frac{y^s ds}{s} \right| \le K \frac{x^a}{(a+c)\log x} \tag{16}$$

where $x > 1$, $a > 0$, and $d > c \ge 0$. The proofs for both can be found in [E1], and we shall not provide them in this paper.

We also need a statement about the density of roots ρ. Namely, there exists H such that for $T \ge H$, the number of roots in the region $T \le \operatorname{Im}(\rho) \le T+1$ is less than $2\log T$. It is clear due to $\xi(s)$ having order of growth 1 that this density must be less than T^ϵ, but to obtain the bound $2\log T$ requires a bit more detail, and it in fact uses Stirling's approximation for the gamma function. We shall not give the proof here, but it can also be found in [E1].

Now, on with the proof that (12) converges to (13). Consider for a given h the differences

$$\sum_\rho \frac{x^\rho}{\rho}\frac{1}{2\pi i}\int_{a-ih}^{a+ih} \frac{x^{s-\rho} ds}{s-\rho} - \sum_{|\operatorname{Im}(\rho)|\le h} \frac{x^\rho}{\rho}\frac{1}{2\pi i}\int_{a-ih}^{a+ih} \frac{x^{s-\rho} ds}{s-\rho} \tag{17}$$

and

$$\sum_{|\operatorname{Im}(\rho)|\le h} \frac{x^\rho}{\rho}\frac{1}{2\pi i}\int_{a-ih}^{a+ih}\frac{x^{s-\rho}ds}{s-\rho} - \sum_{|\operatorname{Im}(\rho)|\le h}\frac{x^\rho}{\rho}. \tag{18}$$

The goal will be to show that both of these are 0, which will prove that (12) is equal to (13), and since the former converges, so does the latter.

We shall consider first an estimate of (17). Write $\rho = \beta + i\gamma$. From (16), we see that the modulus of (17) is at most

$$\begin{aligned}
&\sum_{|\gamma|>h}\left|\frac{x^\rho}{\rho}\right|\left|\frac{1}{2\pi i}\int_{a-ih}^{a+ih}\frac{x^{s-\rho}ds}{s-\rho}\right| \\
&\le 2\sum_{\gamma>h}\frac{x^\beta}{\beta}\left|\frac{1}{2\pi i}\int_{a-\beta+i(\gamma-h)}^{a-\beta+i(\gamma+h)}\frac{x^t dt}{t}\right| \\
&\le 2\sum_{\gamma>h}\frac{x^\beta}{\beta}\cdot K\frac{x^{a-\beta}}{(a-\beta+\gamma-h)\log x} \\
&\le 2K\frac{x^a}{\log x}\sum_{\gamma>h}\frac{1}{\gamma(\gamma-h+c)}
\end{aligned}$$

where $c = a-1>0$ so that $c \le a-\beta$ for all roots ρ. Grouping the with $\gamma > h$ in intervals $h < \gamma \le h+1$, $h+1<\gamma\le h+2$,..., then for large h, the interval $h+j<\gamma<h+j+1$ contains at most $2\log(h+j)$ roots, and thus the modulus of (17) is at most a constant times

$$\sum_{j=0}^{\infty}\frac{\log(h+j)}{(h+j)(j+c)}.$$

This sum obviously converges because of the j^2 in the denominator. However, we need to show that as $h\to\infty$, the sum converges to 0. Choosing h large enough so that $\log(h+j) < (h+j)^{1/2}$ for all $j\ge 0$, and thus the sum is at most

$$\sum_{j=0}^{\infty}\frac{1}{(h+j)^{\frac{1}{2}}(j+c)}$$

which can be made arbitrarily small by choosing large h. Hence (17) goes to 0. Now consider (18). The modulus of (18) is at most

$$2\sum_{0<\gamma\le h}\frac{x^\beta}{\beta}\left|\frac{1}{2\pi i}\int_{a-\beta-i\gamma-ih}^{a-\beta-i\gamma+ih}\frac{x^t dt}{t}\right|.$$

Note the difference in bounds of integration. The integral bounds (15) and (16) imply that this is at most

$$2\sum_{0<\gamma\le h}\frac{x^\beta}{\beta}\left|\frac{1}{2\pi i}\int_{a-\beta-i(h+\gamma)}^{a-\beta+i(h+\gamma)}\frac{x^t dt}{t}\right| + 2\sum_{0<\gamma\le h}\frac{x^\beta}{\beta}\left|\frac{1}{2\pi i}\int_{a-\beta+i(h-\gamma)}^{a-\beta+i(h+\gamma)}\frac{x^t dt}{t}\right|$$
$$\le 2\sum_{0<\gamma\le h}\frac{x^\beta}{\beta}\frac{x^{a-\beta}}{\pi(h+\gamma)\log x} + 2\sum_{0<\gamma\le h}\frac{x^\beta}{\beta}K\frac{x^{a-\beta}}{(a-\beta+h-\gamma)\log x}$$
$$\le \frac{2x^a}{\pi\log x}\sum_{0<\gamma\le h}\frac{1}{\gamma(h+\gamma)} + \frac{2Kx^a}{\log x}\sum_{0<\gamma\le h}\frac{1}{\gamma(c+h-\gamma)},$$

where $c = a-1 > 0$ and $c \le a-\beta$ as before. Now we just need to show tht the two sums

$$\sum_{0<\gamma\le h}\frac{1}{\gamma(h+\gamma)} + \sum_{0<\gamma\le h}\frac{1}{\gamma(c+h-\gamma)}$$

both go to 0. For the first sum, let H be an integer large enough such that the interval $H+j\le\gamma\le H+j+1$ contains at most $2\log(H+j)$ roots. Then

$$\sum_{0<\gamma\le h}\frac{1}{\gamma(h+\gamma)} \le \sum_{0<\gamma\le H}\frac{1}{\gamma(h+\gamma)} + \sum_{0\le j\le h-H}\frac{2\log(H+j)}{(H+j)(h+H+j)},$$

where the first sum has a finite number of terms, and thus goes to 0 as $h\to\infty$. The second sum is at most

$$2\sum_{0\le j\le h-H}(\log h)\left[\frac{1}{h}\left(\frac{1}{H+j}-\frac{1}{h+H+j}\right)\right]$$
$$\le 2\frac{\log h}{h}\sum_{0\le j\le h-H}\frac{1}{H+j}$$
$$\le 2\frac{\log h}{h}\int_{H-1}^{h}\frac{dt}{t}$$
$$\le 2\frac{(\log h)^2}{h},$$

which goes to 0 as $h\to\infty$. A similar calculation shows that the sum

$$\sum_{0<\gamma\le h}\frac{1}{\gamma(c+h-\gamma)}$$

goes to 0 as $h\to\infty$.

With this, we have shown that (17) and (18) go to 0, and hence

$$\lim_{h\to\infty}\sum_{\rho}\frac{x^\rho}{\rho}\frac{1}{2\pi i}\int_{a-ih}^{a+ih}\frac{x^{s-\rho}ds}{s-\rho} - \sum_{|\operatorname{Im}(\rho)|\le h}\frac{x^\rho}{\rho} = 0,$$

and therefore, we have shown the convergence of

$$\sum_{\rho}\frac{x^\rho}{\rho}.$$

6 The Prime Number Theorem and Concluding Remarks

After the argument in the previous section, von Mangoldt then uses a Stieltjes integral to transform the formula for $\psi(x)$ into the formula Riemann obtained for $J(x)$ (the integral is based on $d\psi = (\log x)dJ$). Note that there is no circular reasoning here, as von Mangoldt proved the formula for $\psi(x)$ without using $J(x)$ at all; the plausibility argument at the beginning of section 5.1 using $d\psi = (\log x)dJ$ is purely a result. In the Stieltjes integral that von Mangoldt computed, there were two terms corresponding to the convergent $\sum_\rho x^\rho/\rho$: a first term that contained the sum over ρ but did not contain the variable over which he was integrating, hence the validity of termwise integration, and a second term that contained ρ^2 on the denominator, so that the sum converged uniformly. These formulas can be found on p.63 of [E1].

With these facts, this would constitute an indirect proof that the second term in $J(x)$, i.e. the term $\sum_\rho Li(x^\rho)$, converges. Then the formula

$$J(x) = Li(x) + \sum_\rho Li(x^\rho) - \log 2 + \int_x^\infty \frac{dt}{t(t^2-1)\log t},$$

where $x > 1$ and the second term is summed in order of increasing $|\operatorname{Im}(\rho)|$, is valid.

We now turn our attention for the remainder of the paper to the prime number theorem

$$\pi(x) \sim \frac{x}{\log x},$$

which can almost be seen in Riemann's formula, as $\pi(x) \sim J(x)$, and $x/\log x \sim Li(x)$. Obviously the third and fourth term do not grow, but to show the prime number theorem, one must show that

$$\lim_{x\to\infty} \frac{1}{(x/\log x)} \sum_\rho Li(x^\rho) = 0.$$

Perhaps it is much easier to see this with von Mangoldt's formula after noting that $\psi(x) \sim \pi(x)\log x$. Then the prime number theorem amounts to showing that $\psi(x) \sim x$. From von Mangoldt's formula

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \log(2\pi) + \sum_n \frac{x^{-2n}}{2n},$$

we see that the prime number theorem is equivalent to

$$\lim_{x\to\infty} \frac{-\sum_\rho \frac{x^\rho}{\rho} - \log(2\pi) + \sum_n \frac{x^{-2n}}{2n}}{x} = 0.$$

Since the last 2 terms do not grow with x, it suffices to show that

$$\lim_{x\to\infty} \sum_\rho \frac{x^{\rho-1}}{\rho} = 0,$$

which would follow from $x^{\rho-1} \to 0$ for all ρ. This requires the proof that there are no zeros on the line $\mathrm{Re}(s) = 1$, which is precisely what Hadamard and de la Vallée Poussin showed in their proofs of the prime number theorem.

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