Riemann’s Explicit Formula

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Goal

Our goal is to understand the interesting equation

\[ \pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J\left(\frac{n}{\sqrt{x}}\right), \]

where

\[ J(x) = Li(x) - \sum_{\rho} Li(x^\rho) - \log 2 + \int_x^\infty \frac{dt}{t(t^2 - 1) \log t} \]
Outline

1. Motivation

2. Background

3. Derivation of Riemann’s Explicit Formula

4. Connection Between Riemann’s Formula and the Prime Number Theorem

5. Computations
Prime Number Theorem

Gauss observed in 1792(???)

while he was studying logarithm tables, that the density of
the prime numbers near $x$ is about $1/\log x$.

This led him to conjecture the Prime Number Theorem:

$$\pi(x) \sim \frac{x}{\log x}$$

Proved in 1896 independently by Jacques Hadamard and Charles Jean de la Vallée-Poussin, using complex analysis.
Prime Number Theorem

But how good is the Prime Number Theorem approximation?

$$\pi(x) \sim \frac{x}{\log x}$$
Prime Number Counts

<table>
<thead>
<tr>
<th>$x$</th>
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The integral $\int_0^x \frac{dt}{\log t}$ is called the logarithmic integral, $\text{Li}(x)$

How far off is this approximation?
Is There an Exact Formula for \( \pi(x) \)?
Is There an Exact Formula for \( \pi(x) \)?

Yes.

Riemann found an explicit formula in 1859 in his paper “On the Number of Primes Less Than a Given Magnitude.”

The formula, which is the main result of the paper, was derived largely by (very accurate) intuition.
Background

Recall that

\[ \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}} \]

(Proof: Expand the right-hand side as a series and multiply out all the terms.)

This is the definition of \( \zeta(s) \) for \( \text{Re}(s) > 1 \).
Analytic Continuation

Zeta can be analytically continued to the half-plane $\text{Re}(s) > 0$ by cancelling the pole at $s = 1$. (This was shown in class.)

$$\zeta(s) - \frac{1}{s - 1}$$
Analytic Continuation

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$$\zeta(s) - \frac{1}{s - 1}$$

But zeta can be extended further.
Functional Equation

We will use Riemann’s definition of $\xi(s)$, which has the advantage that it has no poles.

$$\xi(s) = \pi^{-s/2} \Gamma(s/2 + 1)(s - 1)\zeta(s)$$

- The pole of zeta at $s = 1$ is cancelled by $(s - 1)$.
- The poles of $\Gamma(s/2 + 1)$ are cancelled by the trivial zeros of the zeta function, at $s = -2, -4, -6, \text{etc.}$

Thus $\xi(s)$ is an entire function with zeros precisely at the nontrivial zeros of the zeta function.

It satisfies the symmetric relation

$$\xi(s) = \xi(1 - s)$$
Riemann assumed (in 1859) that it is possible to factor an entire function in terms of its roots, obtaining the following expression for $\xi(s)$:

$$\xi(s) = \xi(0) \prod_{\rho} \left( 1 - \frac{s}{\rho} \right)$$

The factorization theorem was not proved until 1893, by Hadamard.
Riemann combined the previous two results

\[ \xi(s) = \pi^{-s/2} \Gamma(s/2 + 1)(s - 1)\zeta(s) \]

\[ \xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \]

to obtain

\[ \zeta(s) = \frac{\xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \pi^{s/2}}{\Gamma(s/2 + 1)(s - 1)} \]

\[ \log \zeta(s) = \log \xi(0) + \sum_{\rho} \log \left(1 - \frac{s}{\rho}\right) + \frac{s}{2} \log \pi - \log \Gamma(s/2 + 1) - \log(s - 1) \]
Define $J(x)$ as a step function as follows:

- $J(0) = 0$
- $J$ jumps by 1 on each prime number.
- $J$ jumps by $\frac{1}{2}$ on each prime square.
- ...
- $J$ jumps by $\frac{1}{n}$ on each prime power $p^n$

$$J(x) = \sum_{p^n \leq x} \frac{1}{n}$$
We want an equation connecting $J(x)$ and $\zeta(s)$. 
Start with the Euler product formula.

\[ \zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}} \]

Take the log of both sides.

\[ \log \zeta(s) = \sum_p \sum_n \frac{1}{n} p^{-ns} \]

From elementary calculus,

\[ p^{-ns} = s \int_{p^n}^{\infty} x^{-s-1} \, dx \]

Substituting, we have

\[ \log \zeta(s) = s \sum_p \sum_n \frac{1}{n} \int_{p^n}^{\infty} x^{-s-1} \, dx \]
\[
\log \zeta(s) = s \sum_p \sum_n \frac{1}{n} \int_{p^n}^\infty x^{-s-1} dx
\]

This converges absolutely, so we can interchange the sums and integral, being careful with the indices:

\[
= s \int_0^\infty \sum_{p^n \leq x} \frac{1}{n} x^{-s-1} dx
\]

\[
= s \int_0^\infty J(x) x^{-s-1} dx
\]
Riemann then applies the inverse Laplace transform (closely related to the inverse Fourier transform):

\[
\frac{\log \zeta(s)}{s} = \int_{0}^{\infty} J(x) x^{-s-1} \, dx
\]

\[
J(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s) \frac{x^s}{s} \, ds
\]
Riemann then applies the inverse Laplace transform (closely related to the inverse Fourier transform):

\[ \frac{\log \zeta(s)}{s} = \int_0^\infty J(x)x^{-s-1} \, dx \]

\[ J(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s) \frac{x^s}{s} \, ds \]

And then substitutes

\[ \log \zeta(s) = \log \xi(0) + \sum_\rho \log \left(1 - \frac{s}{\rho}\right) + \frac{s}{2} \log \pi - \log \Gamma(s/2 + 1) - \log(s - 1) \]
Then a miracle occurs...*
And he obtains

\[ J(x) = Li(x) - \sum_{\rho} Li(x^\rho) - \log 2 + \int_x^\infty \frac{dt}{t(t^2 - 1) \log t} \]

*This was not fully proved until 1895 by von Mangoldt.*

Note: The second term must be summed in order of increasing imaginary part as it converges conditionally.
Counting Primes

Riemann now forms $\pi(x)$ in terms of $J(x)$. The first step is

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots$$

Then he applies a Möbius inversion to obtain

$$\pi(x) = J(x) - \frac{1}{2}J(x^{\frac{1}{2}}) - \frac{1}{3}J(x^{\frac{1}{3}}) + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J\left(\frac{n}{\sqrt{x}}\right),$$
Explicit Formula

Thus he found an exact formula for the prime counting function.

\[ \pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(\sqrt[n]{x}), \]

where

\[ J(x) = Li(x) - \sum_{\rho} Li(x^\rho) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2 - 1) \log t} \]
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$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J\left(\frac{n}{\sqrt{x}}\right),$$

where

$$J(x) = Li(x) - \sum_{\rho} Li(x^\rho) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2 - 1) \log t}$$

- **Main Term**: \(\frac{x}{\log x}\)
- **Oscillatory Term**: \(O(1)\)
- **Log 2 Term**: \(O(1)\)
- **Integral Term**: \(O\left(\frac{1}{x^2}\right)\)
\[ J(x) = \text{Li}(x) - \sum_{\rho} \text{Li}(x^{\rho}) - \log 2 + \int_{x}^{\infty} \frac{dt}{t(t^2 - 1) \log t} \]

- **Main Term**: \( \sim \frac{x}{\log x} \)
- **Oscillatory Term**: \( O(1) \)
- **Log 2 Term**: \( o(1) \)
- **Integral Term**: \( O(1) \)

**Question**
How fast does the oscillatory term grow?
The Oscillatory Term

A deep result assumed by Riemann and proved by von Mangoldt was that

$$\sum_{\rho} Li(x^\rho)$$

actually converges.

But let us see what this term means.
Group the zeros with their conjugate roots $\rho, \bar{\rho}$ so the sum becomes

$$\sum_{\rho} Li(x^\rho) = \sum_{\text{Im}(\rho) > 0} Li(x^\rho) + Li(x^{\bar{\rho}})$$

We use the asymptotic relation

$$Li(x^\rho) \sim \frac{x^\rho}{\rho \log x}$$

to obtain

$$Li(x^\rho) + Li(x^{\bar{\rho}}) \sim \frac{x^\rho}{\rho \log x} + \frac{x^{\bar{\rho}}}{\bar{\rho} \log x} = \frac{x^\sigma}{\log x} \left( \frac{e^{it \log x}}{\sigma + it} + \frac{e^{-it \log x}}{\sigma - it} \right)$$

$$= \frac{x^\sigma}{\log x} \left( \frac{\cos(t \log x) + i \sin(t \log x)}{\sigma + it} + \frac{\cos(t \log x) - i \sin(t \log x)}{\sigma - it} \right)$$

when the roots are written as $\rho = \sigma + it$. 
This simplifies to

\[ \frac{2x^\sigma}{(\sigma^2 + t^2) \log x} \left( \sigma \cos(t \log x) + t \sin(t \log x) \right) \]

Note that \( \sigma^2 + t^2 = |\rho|^2 \), so

\[ \frac{2x^\sigma}{|\rho| \log x} \left( \frac{\sigma}{|\rho|} \cos(t \log x) + \frac{t}{|\rho|} \sin(t \log x) \right) \]

\[ = \frac{2x^\sigma}{|\rho| \log x} \cos(t \log x - \arctan(t/\sigma)) \]

Then

\[ \sum_{\text{Im}(\rho) > 0} Li(x^\rho) + Li(x^{\bar{\rho}}) \sim \sum_{\text{Im}(\rho) > 0} \frac{2x^\sigma}{|\rho| \log x} \cos(t \log x - \arctan(t/\sigma)) \]
This simplifies to

\[
\frac{2x^\sigma}{(\sigma^2 + t^2) \log x} (\sigma \cos(t \log x) + t \sin(t \log x))
\]

Note that \(\sigma^2 + t^2 = |\rho|^2\), so

\[
= \frac{2x^\sigma}{|\rho| \log x} \left( \frac{\sigma}{|\rho|} \cos(t \log x) + \frac{t}{|\rho|} \sin(t \log x) \right)
\]

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= \frac{2x^\sigma}{|\rho| \log x} \cos(t \log x - \arctan(t/\sigma))
\]

Then

\[
\sum_{\text{Im}(\rho) > 0} \mathrm{Li}(x^{\rho}) + \mathrm{Li}(x^{\bar{\rho}}) \sim \sum_{\text{Im}(\rho) > 0} \frac{2x^\sigma}{|\rho| \log x} \cos(t \log x - \arctan(t/\sigma))
\]

If the Riemann Hypothesis is true, then this sum grows like \(O(\sqrt{x} \log x)\). If there are no zeros in a region \(\alpha \leq \sigma < 1\), then the sum obeys \(O(x^{\alpha+\epsilon})\) for \(\epsilon > 0\).
Computation

Suppose we wanted to compute $\pi(10)$. 
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$$\pi(10) = J(10) - \frac{1}{2} J(10^{\frac{1}{2}}) - \frac{1}{3} J(10^{\frac{1}{3}})$$
Computation

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$$\pi(10) = J(10) - \frac{1}{2} J(10^{\frac{1}{2}}) - \frac{1}{3} J(10^{\frac{1}{3}})$$

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<tr>
<td>$J(10)$</td>
<td>6.1656</td>
<td>-.1415</td>
<td>-.6931</td>
<td>.0018</td>
<td>5.3328</td>
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<tr>
<td>$-\frac{1}{2} J(10^{\frac{1}{2}})$</td>
<td>-1.1539</td>
<td>-.1757</td>
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<td>-.0172</td>
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<tr>
<td>$-\frac{1}{3} J(10^{\frac{1}{3}})$</td>
<td>-0.4189</td>
<td>-.1091</td>
<td>.2310</td>
<td>-.0364</td>
<td>-.3334</td>
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<tr>
<td>Total</td>
<td>4.5928</td>
<td>-.4263</td>
<td>-.1155</td>
<td>-.0518</td>
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*5000 iterations of the term in Mathematica 8
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*After the first row, I ignored the log 2 and integral terms in the calculations, as well as all the oscillatory terms after $n = 5$. 
References


